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ON ORTHOGONALLY PEXIDER FUNCTIONAL EQUATION $f(x+y)+g(x-y)=h(x)+k(y)$

SAYED KHALIL EKRAMI*

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
khalil.ekrami@gmail.com, ekrami@pnu.ac.ir

Abstract. One of the pexiderized types of the orthogonally quadratic functional equation is of the form

$$f(x+y)+g(x-y)=h(x)+k(y) \quad (x \perp y).$$

In this paper, we investigate the general solution of this orthogonally pexider functional equation on an orthogonality space in the sense of Rätz, where the function g is odd.

1. Introduction

J. Rätz introduced a defnition of an abstract orthogonality by using four axioms on a real vector space X with $\dim X \geq 2$ (See [3]). Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}$ and $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

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* Speaker.

(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, for any $x \in P$ and any $\lambda \in \mathbb{R}^+$, there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The pair $(X, \perp)$ is called an orthogonality space. Some interesting examples of orthogonality spaces are

(a) Any real vector space X can be made into a orthogonality space with the trivial orthogonality defined on X by (i) for all $x \in X$, $x \perp 0$ and $0 \perp x$, (ii) for all $x, y \in X \setminus \{0\}$, $x \perp y$ if and only if x, y are linearly independent.

(b) Any inner product space $(X, \langle \cdot, \cdot \rangle)$ is an orthogonality space with the ordinary orthogonality given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(c) Any normed space $(X, \|\cdot\|)$ can be made into a orthogonality space with the Birkhoff-James orthogonality defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (a) and (b) are symmetric but example (c) is not.

Let H be an inner product space with $\dim H > 2$ with the usual orthogonality given by $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. Suppose that the functions $f, g, h, k : H \to \mathbb{R}$ satisfy the orthgonally pexider functional equation $f(x + y) + g(x - y) = h(x) + k(y)$ $(x \perp y)$ $\quad (*)$. Fochi [1] showed that the general solution of $(*)$ is of the form

$$\begin{aligned} f(x) &= \tfrac{1}{2}\big(Q(x) + A(x) + B(x) + \phi(\|x\|) + h(0) + k(0)\big),\\ g(x) &= \tfrac{1}{2}\big(Q(x) + A(x) - B(x) - \phi(\|x\|) + h(0) + k(0)\big),\\ h(x) &= Q(x) + A(x) + h(0),\ k(x) = Q(x) + B(x) + k(0), \end{aligned}$$

where $Q : H \to \mathbb{R}$ is a quadratic function, $A, B : H \to \mathbb{R}$ are additive functions and $\phi : [0, \infty) \to \mathbb{R}$ defined by $\phi(\|x\|) = f^e(x) - g^e(x)$ in which f^e and g^e are the even part of f and the even part of g, respectively.

In this paper, let $(X, \perp)$ be an orthogonality space in which $\perp$ is symmetric and Y be a real vector space. We investigate the general solution of $(*)$, where the function g is odd.

2. **The Result**

In this section, we investigate the general solution of $(*)$, where the orthogonality is in the sense of Rätz and the function g is odd.

Lemma 2.1. *Let $(X, \perp)$ be an orthogonality space and Y be a vector space. If the odd function $A : X \to Y$ satisfies the orthgonally functional equation $A(x + y) + A(x - y) = 2A(x)$ $\quad (x \perp y)$, then A is additive.*

Proof. Let $x, y \in X$ with $x \perp y$. Interchanging x with y in $A(x+y)+A(x-y) = 2A(x)$, we get $A(x + y) - A(x - y) = 2A(y)$. By these equations we have $A(x + y) = A(x) + A(y)$. Thus A is orthogonally additive and since A is odd, so on account of Theorem 5 of [3], it is additive. □

Theorem 2.2. *Let $(X, \perp)$ be an orthogonality space, where $\perp$ is symmetric and Y be a vector space. If the functions $f, g, h, k : X \to Y$ satisfy the orthgonally pexider functional equation*

$$x \perp y \quad \Rightarrow \quad f(x+y)+g(x-y)=h(x)+k(y), \tag{2.1}$$

and the function g is odd, then there exist orthogonally quadratic function $Q : X \to Y$ and additive functions $A, B : X \to Y$ such that

$$f(x)=Q(x)+\tfrac{1}{2}\big(A(x)+B(x)\big)+f(0), \quad g(x)=\tfrac{1}{2}\big(A(x)-B(x)\big),$$
$$h(x)=Q(x)+A(x)+h(0), \quad k(x)=Q(x)+B(x)+k(0).$$

Proof. Putting $x=y=0$ in (2.1), we get

$$f(0)=h(0)+k(0). \tag{2.2}$$

Also putting $y=0$ and $x=0$ respectively in (2.1), we get

$$f(x)+g(x)=h(x)+k(0), \tag{2.3}$$

$$f(y)+g(-y)=h(0)+k(y), \tag{2.4}$$

for all $x, y \in X$. Replacing y by $-x$ in (2.4), we have

$$f(-x)+g(x)=h(0)+k(-x). \tag{2.5}$$

From (2.3) and (2.5), we have $f(x)-f(-x)=h(x)-k(-x)+k(0)-h(0)$. Replacing x by $-x$ in the last equation, we get $f(-x)-f(x)=h(-x)-k(x)+k(0)-h(0)$. Using the last two equations, we obtain $h(x)+h(-x)-2h(0)=k(x)+k(-x)-2k(0) \quad (x \in X)$. Define

$$Q(x):=\tfrac{1}{2}\big(h(x)+h(-x)\big)-h(0)=\tfrac{1}{2}\big(k(x)+k(-x)\big)-k(0) \quad (x \in X), \tag{2.6}$$

then Q is an even function and $Q(0)=0$.

Replacing x by $-x$ in (2.1), we get

$$f(-x+y)+g(-x-y)=h(-x)+k(y) \quad (x \perp y). \tag{2.7}$$

From (2.1) and (2.7) (adding and subtracting, respectively), we get

$$f(x+y)+g(x-y)+f(-x+y)+g(-x-y)=2Q(x)+2k(y)+2h(0) \quad (x \perp y), \tag{2.8}$$

$$f(x+y)+g(x-y)-f(-x+y)-g(-x-y)=h(x)-h(-x) \quad (x \perp y). \tag{2.9}$$

Define the function $A : X \to Y$ by $A(x):=\frac{1}{2}\big(h(x)-h(-x)\big)$ $(x \in X)$, then A is an odd function and so $A(0)=0$. Using (2.8) and (2.9), we obtain

$$f(x+y)+g(x-y)=Q(x)+A(x)+h(0)+k(y) \quad (x \perp y),$$

and then by (2.1), we have

$$h(x)=Q(x)+A(x)+h(0) \quad (x \in X). \tag{2.10}$$

Replacing y by $-y$ in (2.1), we get

$$f(x-y)+g(x+y)=h(x)+k(-y) \quad (x \perp y). \tag{2.11}$$

From (2.1) and (2.11) (adding and subtracting, respectively), we get

$$f(x+y)+g(x-y)+f(x-y)+g(x+y)=2h(x)+2Q(y)+2k(0) \quad (x\perp y), \tag{2.12}$$

$$f(x+y)+g(x-y)-f(x-y)-g(x+y)=k(y)-k(-y) \quad (x\perp y). \tag{2.13}$$

Define the function $B: X\to Y$ by $B(x):=\frac{1}{2}\big(k(x)-k(-x)\big)$ $(x\in X)$, then B is an odd function and so $B(0)=0$. Using (2.12) and (2.13), we obtain

$$f(x+y)+g(x-y)=Q(y)+B(y)+h(x)+k(0) \quad (x\perp y),$$

and then by (2.1), we have

$$k(y)=Q(y)+B(y)+k(0) \quad (y\in X). \tag{2.14}$$

Using (2.1), (2.10) and (2.14), we get

$$f(x+y)+g(x-y)=Q(x)+Q(y)+A(x)+B(y)+h(0)+k(0) \quad (x\perp y). \tag{2.15}$$

Putting $y=0$ and $x=0$ respectively in (2.15) and using (2.2), we get

$$f(x)+g(x)=Q(x)+A(x)+f(0) \quad (x\in X),$$
$$f(x)-g(x)=Q(x)+B(x)+f(0) \quad (x\in X).$$

From the last equations, we obtain

$$f(x)=Q(x)+\tfrac{1}{2}\big(A(x)+B(x)\big)+f(0) \quad (x\in X), \tag{2.16}$$

$$g(x)=\tfrac{1}{2}\big(A(x)-B(x)\big) \quad (x\in X). \tag{2.17}$$

It remains to show that Q is orthogonality quadratic and A,B are additive. From (2.16), we have $f(-x)=Q(x)-\frac{1}{2}\big(A(x)+B(x)\big)+f(0)$, and so $f(x)+f(-x)=2Q(x)+2f(0)$ which implies that

$$Q(x)=\tfrac{1}{2}\big(f(x)+f(-x)\big)-f(0) \quad (x\in X). \tag{2.18}$$

Interchanging x by y in (2.15), we have

$$f(x+y)-g(x-y)=Q(x)+Q(y)+A(y)+B(x)+f(0) \quad (x\perp y).$$

Using the last equation and (2.15), we obtain

$$2f(x+y)=2Q(x)+2Q(y)+A(x)+A(y)+B(x)+B(y)+2f(0) \quad (x\perp y),$$

which implies that

$$f(x+y)=Q(x)+Q(y)+\tfrac{1}{2}\big(A(x)+A(y)+B(x)+B(y)\big)+f(0) \quad (x\perp y). \tag{2.19}$$

Let $x, y \in X$ with $x \perp y$, using (2.18) and (2.19), we can conclude that

$$
\begin{aligned}
&Q(x+y)+Q(x-y)\\
&=\tfrac{1}{2}\Big(f(x+y)+f(-x-y)\Big)-f(0)+\tfrac{1}{2}\Big(f(x-y)+f(-x+y)\Big)-f(0)\\
&=\tfrac{1}{2}\Big(f(x+y)+f(-x-y)+f(x-y)+f(-x+y)\Big)-2f(0)\\
&=\tfrac{1}{2}\Big(Q(x)+Q(y)+\tfrac{1}{2}\big(A(x)+A(y)+B(x)+B(y)\big)+f(0)\\
&\quad+Q(x)+Q(y)+\tfrac{1}{2}\big(-A(x)-A(y)-B(x)-B(y)\big)+f(0)\\
&\quad+Q(x)+Q(y)+\tfrac{1}{2}\big(A(x)-A(y)+B(x)-B(y)\big)+f(0)\\
&\quad+Q(x)+Q(y)+\tfrac{1}{2}\big(-A(x)+A(y)-B(x)+B(y)\big)+f(0)\Big)-2f(0)\\
&\quad=2Q(x)+2Q(y).
\end{aligned}
$$

Thus the function Q is orthogonally quadratic. From (2.16) and (2.17), we get

$$A(x)=f(x)+g(x)-Q(x)-f(0) \quad (x \in X). \tag{2.20}$$

Thus for any $x, y \in X$ with $x \perp y$, by (2.15) and (2.20), we get

$$
\begin{aligned}
&A(x+y)+A(x-y)\\
&=f(x+y)+g(x+y)-Q(x+y)-f(0)\\
&\quad+f(x-y)+g(x-y)-Q(x-y)-f(0)\\
&=Q(x)+Q(y)+A(x)+B(y)+f(0)+Q(x)+Q(y)+A(x)-B(y)+f(0)\\
&\quad-Q(x+y)-Q(x-y)-2f(0)=2A(x).
\end{aligned}
$$

Hence by Lemma 2.1, A is additive. This completes the proof. □

3. Conclusion

Let $(X, \perp)$ be an orthogonality space in which $\perp$ is symmetric and Y be a real vector space. In this paper, we investigate the general solution of the orthgonally pexider functional equation $f(x+y)+g(x-y)=h(x)+k(y)$ $(x \perp y)$, where the function g is odd.

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