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ON ORTHOGONALLY ADDITIVE ISOMETRIES

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Abstract. Let H be a real inner product space. In this paper, we show that if a mapping $f : H \to H$ satisfies

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in H$ with $x \perp y$ and

$$\|f(x)\| = \|x\|$$

for all $x, \in H$, then f is an additive mapping.

1. Introduction

There are several orthogonality notions on a real normed space such as Birkhoff-James, isosceles, Phythagorean, Roberts and Diminnie ([3]). J. Rätz [1] introduced an abstract definition of orthogonality on a real vector space by using four axioms. Let us recall the orthogonality in the sense of Rätz.

Definition 1.1. Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

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(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, for any $x \in P$ and any $\lambda \in \mathbb{R}^+$, there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

Some interesting examples of orthogonality spaces are

(a) Any real vector space X can be made into a orthogonality space with the trivial orthogonality defined on X by
   (i) for all $x \in X$, $x \perp 0$ and $0 \perp x$,
   (ii) for all $x, y \in X \setminus \{0\}$, $x \perp y$ if and only if x, y are linearly independent.
(b) Any inner product space $(X, \langle \cdot, \cdot \rangle)$ is an orthogonality space with the ordinary orthogonality given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(c) Any normed space $(X, \| \cdot \|)$ can be made into a orthogonality space with the Birkhoff-James orthogonality defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (a) and (b) are symmetric but example (c) is not. It is remarkable to note that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

Let X be a orthogonality vector space in the sense of Rätz and Y be an abelian group. A function $f : X \to Y$ is called *orthogonally additive*, if $f(x + y) = f(x) + f(y)$ for all $x, y \in X$ with $x \perp y$.

An orthogonally additive mapping can not be additive or linear in general. For example the orthogonally additive mapping $f : H \to \mathbb{R}$ defined on inner product space H by $f(x) = \|x\|^2$ is a quadratic function, since it satisfies the quadratic functional equation

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)$$

for all $x, y \in X$.

Rätz in Corollary 7 of [1] investigated the structure of orthogonally additive mappings and showed that any orthogonally additive mapping f is of the form $a + q$, for a unique additive mapping a and a unique quadratic mapping q.

Moreover he showed that if H is a real inner product space, then any orthogonally additive mapping $f : H \to Y$ is of the form

$$f(x) = a(\|x\|^2) + b(x) \tag{1.1}$$

for all $x \in H$, where $a : \mathbb{R} \to Y$ and $b : H \to Y$ are additive mapping uniquely determined by f. In this paper, we show that any orthogonally additive isometry on an inner product space is an additive mapping.

2. **The result**

Theorem 2.1. *Let H be a real inner product space. If $f : H \to H$ is an orthogonally additive mapping such that*

$$\|f(x)\| = \|x\|$$

for all $x \in H$, then f is an additive mapping.

Proof. Let $\langle .,. \rangle$ denote the inner product of H. It follows from (1.1) that

$$\begin{aligned} \|x\|^2 &= \|f(x)\|^2 \\ &= \langle f(x), f(x) \rangle \\ &= \langle a(\|x\|^2) + b(x), a(\|x\|^2) + b(x) \rangle \\ &= \big\|a(\|x\|^2)\big\|^2 + 2\langle a(\|x\|^2), b(x)\rangle + \|b(x)\|^2 \end{aligned}$$

for all $x \in H$.

Let $r \in \mathbb{Q}$. Then replacing x by rx we get

$$r^2\|x\|^2 = r^4\big\|a(\|x\|^2)\big\|^2 + 2r^3\langle a(\|x\|^2), b(x)\rangle + r^2\|b(x)\|^2 \tag{2.1}$$

for all $x \in H$. Dividing the equation (2.1) by r^4 we have

$$\tfrac{1}{r^2}\|x\|^2 = \big\|a(\|x\|^2)\big\|^2 + 2\tfrac{1}{r}\langle a(\|x\|^2), b(x)\rangle + \tfrac{1}{r^2}\|b(x)\|^2$$

for all $x \in H$. Now taking limit as $r \to \infty$, we get

$$a(\|x\|^2) = 0, \quad \|b(x)\| = \|x\|$$

for all $x \in H$.

For each $t > 0$, put $x = \sqrt{t}\|y\|^{-1}y$ where $0 \neq y \in H$. Then $x \in H$ and

$$a(t) = a(t\|y\|^{-2}\|y\|^2) = a(\big\|\sqrt{t}\|y\|^{-1}y\big\|^2) = a(\|x\|^2) = 0.$$

Thus $a(t) = 0$ for all $t > 0$. Also since a is an additive mapping, so a is odd. Therefore $a(t) = -a(-t) = 0$ for all $t < 0$. This implies that $a = 0$ on $\mathbb{R}$. Thus $f(x) = b(x)$ for all $x \in H$ and f is an additive mapping. □

Proposition 2.2. *Suppose that the functions f, a and b satisfy the equation (1.1) for all $x \in H$. If $a : \mathbb{R} \to H$ and $b : H \to H$ are linear and $f : H \to H$ is bijective, then f is linear.*

Proof. Suppose that $a \neq 0$ on $\mathbb{R}$. Thus for $0 \neq a(1) \in H$, there exists a $0 \neq x_0 \in H$ such that $f(x_0) = -a(1)$. Then we have

$$-a(1) = f(x_0) = f(x) = a(\|x_0\|^2) + b(x_0) = \|x_0\|^2 a(1) + b(x_0).$$

It follows that $(1 + \|x_0\|^2)a(1) = -b(x_0)$ and Then

$$a(1) = b\Big(\frac{-x_0}{1 + \|x_0\|^2}\Big).$$

Therefore

$$f(x) = a(\|x\|^2) + b(x) = \|x\|^2 a(1) + b(x) = \|x\|^2 b\Big(\frac{-x_0}{1 + \|x_0\|^2}\Big) + b(x)$$

for all $x \in H$. So for $x = \frac{1+\|x_0\|^2}{\|x_0\|^2}x_0 \neq 0$ we have

$$\begin{aligned} f\Big(\tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big) &= \Big\|\tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big\|^2 b\Big(\frac{-x_0}{1+\|x_0\|^2}\Big) + b\Big(\tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big) \\ &= b\Big(\Big\|\tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big\|^2 \frac{-x_0}{1+\|x_0\|^2} + \tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big) \\ &= b\Big(-\tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0 + \tfrac{1+\|x_0\|^2}{\|x_0\|^2}x_0\Big) = b(0) = 0. \end{aligned}$$

This contradicts the injectivity of f. Thus $a = 0$ on $\mathbb{R}$ and then $f = b$ is linear. □

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