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# EXISTENCE AND CONVERGENCE OF FIXED POINT RESULTS FOR NONCYCLIC CONTRACTIONS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we study the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space. The presented results extend and improve some recent results in the literature.

## 1. Introduction

Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. A self mapping $T : A\cup B \to A\cup B$ is said to be *noncyclic* provided that $T(A)\subseteq A$ and $T(B)\subseteq B$. We say that $(x,y)\in A\times B$ is an *optimal pair of fixed points* of the noncyclic mapping $T$ provided that

$$Tx=x, \quad Ty=y \quad \text{and} \quad d(x,y)=d(A,B),$$

where $d(A,B)=\inf\{d(a,b):\ a\in A,\ b\in B\}$.

In 2005, Anthony Eldred, Kirk and Veeremani [2] introduced noncyclic mappings and studied the existence of an optimal pair of fixed points of a given mapping.

In 2013, Abkar and Gabeleh [1] introduced noncyclic contraction mappings. As a result of theorem 2.7 of [6], for these mappings, the authors presented the following existence theorem.

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**Theorem 1.1.** *Let $A$ and $B$ be nonempty convex subsets of a uniformly convex Banach space $X$ such that $A$ is closed and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map that is, there exists $c \in [0,1)$ such that*

$$d(Tx, Ty) \le c d(x, y) + (1 - c) d(A, B),$$

*for all $x \in A$ and $y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \ge 0$. Then there exists a unique fixed point $x \in A$ such that $x_n \to x$.*

In this paper, we study the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space.

Here, we recall a definition and fact will be used in the next section.

**Definition 1.2.** [5] A Banach space $X$ is said to be strictly convex if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\left.\begin{array}{c} \|x-p\| \le R \\ \|y-p\| \le R \\ x \ne y \end{array}\right\} \Rightarrow \|\frac{x+y}{2} - p\| < R.$$

**Theorem 1.3.** [6] *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Let $T$ be a noncyclic mapping on $A \cup B$ satisfying*

$$d(Tx, Ty) \le c d(x, y),$$

*for each $x \in A$ and $y \in B$ where $c \in [0,1)$. Then $T$ has a unique fixed point $x$ in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to $x$ for any starting point $x_0 \in A \cup B$.*

## 2. Main results

The following results will be needed to prove the main theorems of this section.

**Lemma 2.1.** *Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$ and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \ge 0$. Then $d(x_n, y_n) \to d(A, B)$ as $n \to \infty$.*

The next two results show the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space.

**Theorem 2.2.** *Let $A$ and $B$ be nonempty weakly closed subsets of a reflexive Banach space $X$ and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

*Proof.* If $d(A, B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for $y_0 \in A$, define $y_{n+1} := Ty_n$ for each $n \ge 0$. By Lemma 2.2 of [6], the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. As $X$ is reflexive and $A$ is weakly closed, the sequence $\{x_n\}$ has a subsequence$\{x_{n_k}\}$ with $x_{n_k} \xrightarrow{w} x \in A$. As $\{y_{n_k}\}$ is bounded and $B$ is weakly closed, we can say, without loss of generality, that $y_{n_k} \xrightarrow{w} y \in B$

as $k \to \infty$. Since $x_{n_k} - y_{n_k} \xrightarrow{w} x - y \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x-y) = \|x-y\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - y_{n_k})| \leq \|f\| \|x_{n_k} - y_{n_k}\| = \|x_{n_k} - y_{n_k}\|.$$

Since

$$\lim_{k\to\infty} f(x_{n_k} - y_{n_k}) = f(x-y) = \|x-y\|,$$

it follows from Lemma 2.1 that

$$\|x-y\| = \lim_{k\to\infty} |f(x_{n_k} - y_{n_k})| \leq \lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = d(A,B).$$

Thus $\|x-y\| = d(A,B)$. □

**Definition 2.3.** [4] A mapping $F : C \subseteq X \to X$ is called demiclosed at $y$ if, whenever $x_n \xrightarrow{w} x \in C$ and $Fx_n \xrightarrow{s} y \in X$, it follows that $Fx = y$.

Let $I$ is the identity map, $I - T : A \cup B \to X$ is demiclosed at 0 if whenever $x_n$ is a sequence in $A \cup B$ such that $x_{n_k} \xrightarrow{w} x \in A \cup B$ and $(I-T)x_n \xrightarrow{s} 0$ as $n \to \infty$, then $(I-T)x = 0$.

**Theorem 2.4.** *Let $A$ and $B$ be nonempty subsets of a reflexive Banach space $X$ such that $A$ is weakly closed and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. Then there exists $x \in A$ such that $Tx = x$ provided one of the following conditions is satisfied:*

(a) *$T$ is weakly continuous on $A$;*
(b) *$I - T : A \cup B \to X$ is demiclosed at 0.*

*Proof.* If $d(A,B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A,B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. By Lemma 2.2 of [6], the sequence $\{x_n\}$ is bounded. As $X$ is reflexive and $A$ is weakly closed, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \xrightarrow{w} x \in A$ as $k \to \infty$.

(a) Since $T$ is weakly continuous on $A$ and $T(A) \subseteq A$, we have $x_{n_k+1} \xrightarrow{w} Tx \in A$ as $k \to \infty$. Thus $x_{n_k} - x_{n_k+1} \xrightarrow{w} x - Tx$ as $k \to \infty$. We assume the contrary, $x - Tx \neq 0$. Since $x_{n_k} - x_{n_k+1} \xrightarrow{w} x - Tx \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x - Tx) = \|x - Tx\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - x_{n_k+1})| \leq \|f\| \|x_{n_k} - x_{n_k+1}\| = \|x_{n_k} - x_{n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{n_k} - x_{n_k+1}) = f(x - Tx) = \|x - Tx\|,$$

it follows from Lemma 2.1 that

$$\|x - Tx\| = \lim_{k\to\infty} |f(x_{n_k} - x_{n_k+1})| \leq \lim_{k\to\infty} \|x_{n_k} - x_{n_k+1}\| = 0.$$

Thus $\|x - Tx\| = 0$, a contradiction.

(b) By Lemma 2.1, we have

$$\|x_{n_k} - Tx_{n_k}\| = \|x_{n_k} - x_{n_k+1}\| \to 0$$

as $k \to \infty$. So $(I - T)x_{n_k} \xrightarrow{s} 0$ as $k \to \infty$. As $I - T : A \cup B \to X$ is demiclosed at 0, it follows that $(I - T)x = 0$. Hence $Tx = x$. □

The next result show the existence and uniqueness of a best proximity point for a cyclic contraction map in a reflexive and strictly Banach space. This theorem guarantees the uniqueness in Theorem 3.5 of [3].

**Theorem 2.5.** *Let $A$ and $B$ be nonempty closed and convex subsets of a reflexive and strictly convex Banach space $X$ and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. If $(A - A) \cap (B - B) = \{0\}$, then there exists a unique optimal pair of fixed points $(x, y) \in A \times B$ for $T$.*

*Proof.* If $d(A, B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A, B) > 0$. Since $A$ is closed and convex, it is weakly closed. It follows from Theorem 2.2 that there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$. To show the uniqueness of $(x, y)$, suppose that there exists another $(x', y') \in A \times B$ such that $\|x' - y'\| = d(A, B)$. As $(A - A) \cap (B - B) = \{0\}$ we conclude that $x - x' \neq y - y'$ and so $x - y \neq x' - y'$ . Since $A$ and $B$ are both convex, it follows from the strict convexity of $X$ that

$$\|\frac{x + x'}{2} - \frac{y + y'}{2}\| = \|\frac{x - y + x' - y'}{2} - 0\| < d(A, B),$$

a contradiction. As

$$\|Tx - Ty\| = \|x - y\| = d(A, B),$$

we conclude, from the uniqueness of $(x, y)$, that $(Tx, Ty) = (x, y)$. Thus $Tx = x$ and $Ty = y$. □

## References


1. A. Abkar, and M. Gabeleh, *Proximal quasi-normal structure and a best proximity point theorem*, J. Nonlinear Convex Anal., **14** (4) (2013) 653-659.
2. A. Anthony Eldred, W. A. Kirk, and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math., **171** (3) (2005) 283-293.
3. A. Fernández-León, and M. Gabeleh, *Best proximity pair theorems for noncyclic mappings in Banach and metric spaces.* Fixed Point Theory, 17(1) (2016), 63-84.
4. T. M. Gallagher, *The demiclosedness principle for mean nonexpansive mappings*, J. Math. Anal. Appl., **439** (2) (2016) 832-842.
5. K. Goebel, and W.A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, 1990.
6. A. Safari-Hafshejani, A. Amini-Harandi, and M. Fakhar, *Best proximity points and fixed points results for non-cyclic and cyclic Fisher quasi-contractions*, Numer. Funct. Anal. Optim., **40** (5) (2019) 603-619.