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# EXISTENCE AND CONVERGENCE OF BEST PROXIMITY POINT RESULTS FOR CYCLIC QUASI-CONTRACTIONS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we study the existence of a best proximity point for a cyclic quasi-contraction map in a reflexive Banach space. The presented results extend and improve some recent results in the literature.

## 1. Introduction

In 2009 Al-Thagafi and Shahzad [1] prove the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space.

**Theorem 1.1.** [1, Theorem 9] *Let $A$ and $B$ be nonempty weakly closed subsets of a reflexive Banach space $X$ and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

**Definition 1.2.** [4] Let $A$ and $B$ be nonempty subsets of a normed space $X$ and $T$ be a cyclic map on $A \cup B$. We say that $T$ satisfies the proximal property if

$$x_n \xrightarrow{w} x \in A \cup B \quad \textit{and} \quad \|x_{n_k} - Tx_{n_k}\| \to d(A, B) \Longrightarrow \|x - Tx\| = d(A, B).$$

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**Theorem 1.3.** [1, Theorem 10] *Let $A$ and $B$ be nonempty subsets of a reflexive Banach space $X$ such that $A$ is weakly closed and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *$T$ is weakly continuous on $A$;*
(b) *$T$ satisfies the proximal property.*

**Theorem 1.4.** [1, Theorem 12] *Let $A$ and $B$ be nonempty subsets of a reflexive and strictly convex Banach space $X$ such that $A$ is closed and convex and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *$T$ is weakly continuous on $A$;*
(b) *$T$ satisfies the proximal property.*

**Theorem 1.5.** [6] *Let $A$ and $B$ be nonempty and closed subsets of a complete metric space $(X, d)$. Let $T$ be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\},$$

*for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then $T$ has a unique fixed point $x$ in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to $x$ for any starting point $x_0 \in A \cup B$.*

## 2. MAIN RESULTS

The following results will be needed to prove the main theorems of this section.

**Lemma 2.1.** *Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$ and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map, that is there exists $\lambda \in [0, 1)$ such that*

$$d(Ty, Tx) \leq \lambda max\{d(x, y), d(x, Tx), d(Ty, y)\} + (1 - \lambda)d(A, B)$$

*for all $x \in A$ and $y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $d(x_{2n}, x_{2n+1}) \to d(A, B)$ as $n \to \infty$.*

The next two results that are extentions of Theorems 1.1 and 1.3, show the existence of a best proximity point for a cyclic qusi-contraction map in a reflexive Banach space.

**Theorem 2.2.** *Let $A$ and $B$ be nonempty weakly closed subsets of a reflexive Banach space $X$ and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

*Proof.* If $d(A, B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for each $n \geq 0$. By Lemma 3.2 of [3], the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. As $X$ is reflexive and $A$ is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence$\{x_{2n_k}\}$ with $x_{2n_k} \xrightarrow{w} x \in A$. As $\{x_{2n_k+1}\}$ is bounded and $B$ is weakly closed, we can say, without loss of generality, that $x_{2n_k+1} \xrightarrow{w} y \in B$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - y \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad \textit{and} \quad f(x - y) = \|x - y\|.$$

For each $k \geq 1$, we have

$$|f(x_{2n_k} - x_{2n_k+1})| \leq \|f\| \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{2n_k} - x_{2n_k+1}) = f(x - y) = \|x - y\|,$$

it follows from Lemma 2.1 that

$$\|x - y\| = \lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = d(A, B).$$

Thus $\|x - y\| = d(A, B)$. □

The following theorem is proved in a completely similar way to the proof of theorem 1.3.

**Theorem 2.3.** *Let $A$ and $B$ be nonempty subsets of a reflexive Banach space $X$ such that $A$ is weakly closed and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *$T$ is weakly continuous on $A$.*
(b) *$T$ satisfies the proximal property.*

*Proof.* If $d(A, B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. By Lemma 3.2 of [3], the sequence $\{x_{2n}\}$ is bounded. As $X$ is reflexive and $A$ is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ with $x_{2n_k} \xrightarrow{w} x \in A$ as $k \to \infty$.

(a) Since $T$ is weakly continuous on $A$ and $T(A) \subseteq B$, we have $x_{2n_k+1} \xrightarrow{w} Tx \in B$ as $k \to \infty$. Thus $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - Tx$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - Tx \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad \textit{and} \quad f(x - Tx) = \|x - Tx\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - x_{n_k+1})| \leq \|f\| \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{2n_k} - x_{2n_k+1}) = f(x - Tx) = \|x - Tx\|,$$

it follows from Lemma 2.1 that

$$\|x - Tx\| = \lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = d(A, B).$$

Thus $\|x - Tx\| = d(A, B)$.

(b) By Lemma 2.1 , we have

$$\|x_{2n_k} - Tx_{2n_k}\| = \|x_{2n_k} - x_{2n_k+1}\| \to d(A, B).$$

as $k \to \infty$. As $T$ satisfies the proximal property, it follows that $\|x - Tx\| = d(A, B)$. □

The next result that is extention of Theorem 1.4,shows the existence and uniqueness of a best proximity point for a cyclic quasi contraction map in a reflexive and strictly Banach space.

**Theorem 2.4.** *Let $A$ and $B$ be nonempty subsets of a reflexive and strictly convex Banach space $X$ such that $A$ is closed and convex let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *$T$ is weakly continuous on $A$.*
(b) *$T$ satisfies the proximal property.*

*Proof.* If $d(A, B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A, B) > 0$. Since $A$ is closed and convex, it is weakly closed. It follows from Theorem 2.3 that there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$. Also

$$\begin{aligned} \|T^2x - Tx\| &\leq \lambda \max\left\{\|Tx - x\|, \|T^2x - Tx\|\right\} + (1 - \lambda)d(A, B) \\ &= \lambda\|T^2x - Tx\| + (1 - \lambda)d(A, B), \end{aligned}$$

and so $\|T^2x - Tx\| = d(A, B)$. In fact, $T^2x = x$. To see this, assume that $T^2x \neq x$. It follows, from the convexity of $A$ and the strict convexity of $X$, that

$$\|\frac{T^2x + x}{2} - Tx\| < d(A, B),$$

a contradiction. A similar argument shows the uniqueness of x follows as □

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