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QUANTUM DETECTION PROBLEM VIA FRAME THEORY

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ABSTRACT. One of the major problems in quantum detection problem is the injectivity problem which studied via frame theory. It was shown that a quantum injective frame is a frame that can be used to distinguish density operators (states) from their frame measurements, and the frame quantum detection problem asks to characterize all such frames. In this note, we will answer to some of the problems in quantum detection using g-frames.

1. INTRODUCTION

1.1. **Frames and g-frames in Hilbert spaces.** Frames in Hilbert spaces were first introduced by Duffin and Schaeffer to deal with nonharmonic Fourier series in 1952 [4] and widely studied from 1986 since the great work by Daubechies, Grossmann and Meyer. Frames are basis- like building blocks that span a vector space but allow for linear dependency, which is useful to reduce noise, find sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis etc.

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space, $\{\mathcal{H}\}_{i\in\mathbb{I}}$ a family of Hilbert spaces, $\mathbb{I}$ a finite or countable index set, $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$ and $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis (ONB) for $\mathcal{H}$. The operator T is a Hilbert-Schmidt operator if $\|T\|_2 := (\sum_i \|Te_i\|^2)^{\frac{1}{2}} < \infty$ and it is trace class operator if $\|T\|_1 := \sum_i \langle |T|e_i, e_i\rangle < \infty$, in this case

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the trace of T is $tr(T) = \sum_i \langle Te_i, e_i\rangle$ which is finite and independent of the orthonormal basis. It is known that trace induces an inner product by $\langle T, S\rangle_{HS} = tr(TS^*)$ for the Hilbert -Schmidt operators T, S. Let $Sym(\mathcal{H}) := \{T : T \in \mathcal{B}(\mathcal{H}), T = T^*\}$ denote the real Banach space of self-adjoint operators on $\mathcal{H}$ and $Sym^+(\mathcal{H}) := \{T \in Sym(\mathcal{H}), T = T^* \geq 0\}$. The family of trace class operators on $\mathcal{H}$ denoted by $\mathcal{S}_1$ and Hilbert-Schmidt by $\mathcal{S}_2$.

Definition 1.1. A countable family of elements $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that:

$$A\|x\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}. \tag{1.1}$$

The numbers A and B are celled the lower and upper frame bounds, respectively. The frame $\{f_i\}_{i\in\mathbb{I}}$ is called tight, if $A = B$ and is called Parseval, if $A = B = 1$. Also the sequence $\{f_i\}_{i\in\mathbb{I}}$ is called Bessel sequence, if the upper inequality in (1.1) holds. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ve_i\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and the operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded and invertible operator.

For the Bessel sequence $\{f_i\}_{i\in\mathbb{I}}$, the analysis operator $T : \ell^2(\mathbb{I}) \longrightarrow \mathcal{H}$ is defined by $T(x) = \{\langle x, f_i\rangle\}_{i\in\mathbb{I}}$, for all $x \in \mathcal{H}$. Its adjoint operator $T^* : \mathcal{H} \to \ell^2(\mathbb{I})$ called synthesis operator and $T\{c_i\} = \sum_{i\in\mathbb{I}} c_i f_i$, for all $\{c_i\} \in \ell^2$. The operator $S : \mathcal{H} \longrightarrow \mathcal{H}$, which is defined by $S(x) = TT^*(x) = \sum_{i\in\mathbb{I}} \langle x, f_i\rangle f_i$, for all $x \in \mathcal{H}$, is called the frame operator. For a frame $\{f_i\}_{i\in\mathbb{I}}$, the operator T^* is onto, T is one to one and S is positive, self adjoint and invertible.

A dual of the Bessel sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ is a Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$, such that

$$x = \sum_{i\in\mathbb{I}} \langle x, g_i\rangle f_i, \quad x \in \mathcal{H}.$$

For a more complete treatment of frame theory we recommend the excellent book of Christensen [5]. Over the years, various extensions of the frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames that was introduced by W. Sun in [6]. For example, one can consider: bounded quasi-projectors, fusion frames, pseudo-frames, oblique frames, outer frames and etc.

Definition 1.2. We call $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A = B$ and we call it a Parseval g-frame if $A = B = 1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathbb{I}} \Lambda_i^*\Lambda_i f, \quad f \in \mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \leq S_\Lambda \leq BI$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in I} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in I} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ with bounds $\frac{1}{B}$, $\frac{1}{A}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ and $\{\Lambda_i\}_{i\in\mathbb{I}}$ are dual g-frames with respect to each other.

1.2. **Positive Operator-Valued Measures (POVM) and Quantum Detection Problem.** Quantum theory tries to predict the probability of observing outcomes from a sequence of measurements of the system in an unknown state. This process is called quantum state tomography. The outcome statistics are described by a positive operator-valued measure (POVM). In fact, quantum measurement extracts the transmitted information from received quantum signals and therefore performs an important role of quantum communications, [7]. The simplest quantum measurement is the projection-valued measure (PVM), also called standard measurement or von Neumann measurement, where elementary projectors are usually used. Sometimes the positive-operator valued measure (POVM) is more efficient of obtaining information about the state of a quantum system than a standard measurement. Let X denote a set of outcomes (e.g. this could be a finite or infinite subset of $\mathbb{Z}^d$ or $\mathbb{R}^d$). Let β denote a sigma algebra of subsets of X.

Definition 1.3. A positive operator-valued measure (POVM) is a function $\Pi : \beta \to Sym^+(\mathcal{H})$ satisfying:

1. $\Pi(\phi) = 0$.
2. For every at most countable disjoint family $\{V_i\} \subset \beta$ and $x, y \in \mathcal{H}$ we have $\langle\Pi(\bigcup V_i)x, y\rangle = \sum_i \langle\Pi(V_i)x, y\rangle$.
3. $\Pi(X) = I$ (the identity operator).

A quantum system is defined as a von Neumann algebra $\mathcal{A}$ of operators acting on $\mathcal{H}$. The set of states on $\mathcal{H}$ is $\mathbb{S}(\mathcal{H}) := \{T \in \mathcal{S}_1, T = T^* \geq 0, tr(T) = 1\}$. It is known that [1], the set of quantum states are in one-to-one correspondence with the linear functionals on $\mathcal{A}$ of the form:

$$\rho : \mathcal{A} \to \mathbb{C}, \quad for \quad S \in \mathbb{S}(\mathcal{H}), \rho(T) = tr(ST), \forall T \in \mathcal{A}.$$

2. Injectivity Problem via frames

The quantum detection problem with discrete frame coefficient measurements was recently settled by Botelho-Andrade et al. for both finite and infinite dimensional Hilbert spaces in [1, 2], where the characterization was given in terms the spanning properties of some derived sequences from the frame vectors. It is natural to study this problem and results for frame's extensions. For continuous frames it has been investigated by Deguang Han and et.al. [3]. In this note, we will study for g-frames which follows the similar results for fusion frames.

Definition 2.1. A family of vectors $\mathcal{X} = \{x_k\}_{k\in\mathbb{I}}$ in a Hilbert space $\mathcal{H}$ is said to be injective if given a Hilbert- Schmidt self-adjoint operator T satisfying $\langle Tx_k, x_k\rangle = 0$ for all $k \in \mathbb{I}$, then $T = 0$.

It is known [1] that if a family of vectors gives injectivity in a Hilbert space $\mathcal{H}^n$, then it is a frame for $\mathcal{H}^n$ and in case $\{x_k\}_{k\in\mathbb{I}}$ a frame for $\mathcal{H}$ which gives injectivity. If F is a bounded invertible operator on $\mathcal{H}$, then $\{Fx_k\}_{k\in\mathbb{I}}$ also gives injectivity.

Definition 2.2. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$, where the bounded operators $\Lambda_i, i \in \mathbb{I}$ are Hilbert-Schmidt operators. Λ is said to be injective if given a Hilbert- Schmidt self-adjoint operator T satisfying $\langle \Lambda_i T, \Lambda_i\rangle_{\mathcal{HS}} = 0$ for all $k \in \mathbb{I}$, then $T = 0$.

Theorem 2.3. *Let $\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ be g-frame for $\mathcal{H}$. The following statements are equivalent.*

(1) *Whenever T_1, T_2 are Hilbert-Schmidt, positive, self adjoint and $\langle \Lambda_i T_1, \Lambda_i\rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i\rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.*
(2) *Whenever T_1, T_2 are Hilbert-Schmidt, self adjoint and $\langle \Lambda_i T_1, \Lambda_i\rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i\rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.*
(3) *$\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ is injective.*

References

1. S. Botelho-Andrade, P. G. Casazza, D. Cheng and Tin T. Tran, *The Solution to the Frame Quantum Detection Problem*, J. Fourier. Anal. Appl. **25**(2019) 2268-2323.
2. S. Botelho-Andrade, P. G. Casazza, D. Cheng and Tin T. Tran, *The quantum detection problem: a survey.* Springer Proc. Math. Stat. **255**(2017) 337-352.
3. D. Han, Q. Hu and R. Liu, *Injective continuous frames and quantum detections*, Banach J. Math. Anal. **15**:12(2021) https://doi.org/10.1007/s43037-020-00086-7
4. R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, J. Trans. Amer. Math. Soc., **72**(1952) 341-366 .
5. O. Christensen, *An introduction to frames and Riesz Basis,* Birkhäuser, Boston, 2016.
6. W. Sun, *G-frames and g-Riesz bases*, *J. Math. Anal. Appl.*, **322(1)**, (2006) 437-452.
7. M. Paris and J. Eehacek (Eds) *Quantum state estimation*, Springer-Verlag, Berlin , 2004.