The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

A NOTE ON PHASE (NORM) RETRIEVABLE REAL HILBERT SPACE FUSION FRAMES

F. AKRAMI AND A. RAHIMI

Department of Mathematics, University of Maragheh, Maragheh, Iran
fateh.akrami@gmail.com
rahimi@maragheh.ac.ir

Abstract. In this manuscript, we will present several new results in finite and countable dimensional separable real Hilbert space phase retrieval and norm retrieval by fusion frames. We will characterize of norm retrieval for fusion frames similar norm retrieval for vectors and we will show that only one direction holds for fusion frames. Similar vector case we will show that every tight fusion frame can do norm retrieval. Also we will show that the unitary operators preserve phase (norm) retrievability of fusion frames. We will provide numerous examples to show that our results are best possible.

1. Introduction

Fusion frames are an emerging topic of frame theory, with applications to communications and distributed processing. Fusion frames were introduced by Casazza and Kutyniok in [7] and further developed in their joint paper [8] with Li. The theory for fusion frames is available in arbitrary separable Hilbert spaces (finite dimensional or not).

We first give the background material needed for the paper. Let $\mathbb{H}$ be finite or infinite dimensional separable real Hilbert space and $B(\mathbb{H})$ be the class of all bounded linear operators defined on $\mathbb{H}$. The natural numbers and real numbers are denoted by "$\mathbb{N}$" and "$\mathbb{R}$", respectively. We use $[m]$ instead of the set $\{1, 2, 3, \dots, m\}$ and use $[\{f_i\}_{i\in I}]$ instead of $span\{f_i\}_{i\in I}$ where I

2010 *Mathematics Subject Classification.* 42C15, 42C40.
Key words and phrases. Fusion frame, Phase retrieval, Norm retrieval.

is a finite or infinite subset of $\mathbb{N}$. We denote by $\mathbb{R}^n$ a n dimensional real Hilbert space. We start with the definition of a real Hilbert space frame.

Definition 1.1. A family of vectors $\{f_i\}_{i\in I}$ in a finite or infinite dimensional separable Hilbert space $\mathbb{H}$ is a **frame** if there are constants $0 < A \leq B < \infty$ so that $A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2$, for all $f \in \mathbb{H}$. The constants A and B are called the lower and upper frame bounds for $\{f_i\}_{i\in I}$, respectively. If only an upper frame bound exists, then $\{f_i\}_{i\in I}$ is called a **B-Bessel set** or simply **Bessel** when the constant is implicit. If $A = B$, it is called an **A-tight frame** and in case $A = B = 1$, it called a **Parseval frame**. The values $\{\langle f, f_i\rangle\}_{i=1}^{\infty}$ are called the frame coefficients of the vector $f \in \mathbb{H}$.

Throughout the paper, $\{e_i\}_{i=1}^{n}$ will be used to denote the canonical basis for the real space $\mathbb{R}^n$, i.e., a basis for which $\langle e_i, e_j\rangle = \delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Definition 1.2. A family of vectors $\{f_i\}_{i\in I}$ in a real Hilbert space $\mathbb{H}$ does **phase (norm) retrieval** if whenever $x, y \in \mathbb{H}$, satisfy
$|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$ for all $i \in I$, then $x = \pm y$ ($\|x\| = \|y\|$).

Phase retrieval was introduced in reference [2] . See reference [1] for an introduction to norm retrieval.

Note that if $\{f_i\}_{i\in I}$ does phase (norm) retrieval, then so does $\{a_i f_i\}_{i\in I}$ for any $0 < a_i < \infty$ for all $i \in I$. But in the case where $|I| = \infty$, we have to be careful to maintain frame bounds. This always works if $0 < \inf_{i\in I} a_i \leq sup_{i\in I} a_i < \infty$. But this is not necessary in general.

The complement property is an essential issue here. Since in the finite dimensional setting, frames are equivalent with spanning sets, We just give the complement property in the finite case from [5].

Definition 1.3. A family of vectors $\{f_k\}_{k=1}^{m}$ in $\mathbb{R}^n$ has the **complement property** if for any subset $I \subset [m]$,

$$either \quad span\{f_k\}_{k\in I} = \mathbb{R}^n \quad \text{or} \quad span\{f_k\}_{k\in I^c} = \mathbb{R}^n.$$

The following result first appeared in [2]. This result is valid for infinite case too.

Theorem 1.4. *A family of vectors $\{f_i\}_{i=1}^{m}$ in $\mathbb{R}^n$ does phase retrieval if and only if it has the complement property.*

2. Phase (norm) retrievable Fusion frames

Here $\{W_i\}_{i\in I}$ is a family of closed subspaces of $\mathbb{H}$ and $\{v_i\}_{i\in I}$ is a family of positive weights. Also we denote by P_i the orthogonal projection onto W_i.

Definition 2.1. A family $\{(W_i, v_i)\}_{i\in I}$ with W_i subspaces of $\mathbb{H}$, v_i weights, and P_i the projection onto W_i, is a **fusion frame** for $\mathbb{H}$ if there exist constants $A, B > 0$ such that $A\|f\|^2 \leq \sum_{i\in I} v_i^2\|P_i f\|^2 \leq B\|f\|^2$, for all $f \in \mathbb{H}$.

The constants A and B are called the **fusion frames bounds**. We also refer to the fusion frames as $\{P_i, v_i\}_{i\in I}$ or just $\{P_i\}_{i\in I}$ if the weights are all one.

For more details on fusion frames, we recommend [7]. Improving and extending the notions of phase and norm retrievability, we present the definition of phase (norm) retrievable to fusion frames.

Definition 2.2. A family of projections $\{P_i\}_{i\in I}$ in a real Hilbert space $\mathbb{H}$ does **phase (norm) retrieval** if whenever $x, y \in \mathbb{H}$, satisfy $\|P_i x\| = \|P_i y\|$ for all $i \in I$, then $x = \pm y$ ($\|x\| = \|y\|$).

A fusion frame $\{(W_i, v_i)\}_{i\in I}$ is phase (norm) retrievable for $\mathbb{H}$ if and only if the family of projections $\{P_i\}_{i\in I}$ is phase (norm) retrievable for $\mathbb{H}$, where $P_i = P_{W_i}$ is the orthogonal projection onto W_i $(i \in I)$.

One part of the importance of fusion frames is that it is both necessary and sufficient to be able to string together frames for each of the subspaces W_k (with uniformly bounded frame constants) to get a frame for $\mathbb{H}$ which is proved in [4]:

Theorem 2.3. *Let $\{W_i\}_{i\in I}$ be subspaces of $\mathbb{R}^n$. The following are equivalent:*

1. *$\{W_i\}_{i\in I}$ is phase retrievable.*
2. *For every orthonormal basis $\{f_{ij}\}_{j\in I_i}$ for W_i, the family $\{f_{ij}\}_{j\in I_i, i\in I}$ does phase retrieval.*

We note that (2) of the above theorem must hold for **every** orthonormal basis for the subspaces. For example, let $\{\phi_i\}_{i=1}^3$ and $\{\psi_i\}_{i=1}^3$ be orthonormal bases for $\mathbb{R}^3$ so that $\{\phi_i\}_{i=1}^3 \cup \{\psi_i\}_{i=1}^3$ is full spark. Let

$$W_1 = [\phi_1] \quad W_2 = [\phi_2] \quad W_3 = [\phi_3] \quad W_4 = [\psi_1, \psi_2].$$

Then $\{W_i\}_{i=1}^4$ is a fusion frame for $\mathbb{R}^3$ and $\{\phi_1, \phi_2, \phi_3, \psi_1, \psi_2\}$ is full spark and so does phase retrieval for $\mathbb{R}^3$. But it is known that 4 subspaces of $\mathbb{R}^3$ cannot do phase retrieval [4].

The corresponding result for norm retrieval does not make sense, because every orthonormal basis for a subspace does norm retrieval.

We can strengthen this theorem.

Theorem 2.4. *Let $\{(W_k, v_k)\}_{k\in I}$ be a phase (norm) retrievable fusion frame for $\mathbb{H}$ and $\{f_{ij}\}_{j\in I_i}$ be a norm retrievable frame for W_i for $i \in I$. Then $\{v_i f_{ij}\}_{j\in I_i, i\in I}$ is a phase (norm) retrievable frame for $\mathbb{H}$.*

The following theorem shows that the unitary operators preserve phase (norm) retrievability of fusion frames.

Theorem 2.5. *Let $\{(W_i, v_i)\}_{i\in I}$ be a phase (norm) retrievable fusion frame for $\mathbb{H}$. If $T \in B(\mathbb{H})$ is a unitary operator, then $\{(TW_i, v_i)\}_{i\in I}$ is also a phase (norm) retrievable fusion frame.*

Theorem 2.6. *Let $\{(W_i, v_i)\}_{i\in I}$ be a norm retrievable fusion frame for a Hilbert space $\mathbb{H}$, with projections $\{P_i\}_{i\in I}$. Let $\{Q_i\}_{i\in I}$ be projections from W_i to W_i, and let $W_i' = Q_iW_i$ and $W_i'' = (I - Q_i)W_i$ for all $i \in I$. Then $\{(W_i', v_i)\}_{i\in I} \bigcup \{(W_i'', v_i)\}_{i\in I}$ is a norm retrievable fusion frame for $\mathbb{H}$.*

The characterization of norm retrievable families of vectors first appeared in [10].

Theorem 2.7. *A family of vectors $\{f_k\}_{k=1}^{\infty}$ does norm retrieval for $\mathbb{H}$ if and only if for any subset $I \subset \mathbb{N}$, $(\overline{span}\{f_k\}_{k\in I})^{\perp} \perp (\overline{span}\{f_k\}_{k\in I^c})^{\perp}$.*

One direction of this implication holds for fusion frames.

Theorem 2.8. *Let$\{W_i, v_i\}_{i\in I}$ be a fusion frame in $\mathbb{R}^n$. If $\{W_i, v_i\}_{i\in I}$ does norm retrieval, then whenever $J \subset I$, and $x \perp W_j$ for all $j \in J$ and $y \perp W_j$ for all $j \in J^c$, then $x \perp y$.*

In contrast to the vector case, the converse of the above theorem fails in general.

Theorem 2.9. *If $\{P_i, v_i\}_{i=1}^{m}$ is an A-tight fusion frame, then it does norm retrieval.*

Theorem 2.10. *Let $\{e_i\}_{i=1}^{n}$ be the canonical orthonormal basis for $\mathbb{R}^n$. Let $\{I_i\}_{i=1}^{m}$ be subsets of $[n]$. Let $W_i = span\{e_j\}_{j\in I_i}$, for all $i \in [m]$. Also let $J_i = (x_1, x_2, \ldots, x_n)$ where $x_k = 1$ if $k \in I_i$ and $x_k = 0$ otherwise. Assume there exists a natural number K and $\epsilon_i = \pm 1$ so that $\sum_{i=1}^{m} \epsilon_i J_i = K(1, 1, \ldots, 1) \in \mathbb{R}^n$. Then $\{W_i, v_i\}_{i=1}^{m}$ does norm retrieval for all $0 < v_i < \infty$.*

References

1. F. Akrami, P. G. Casazza, M. A. Hasankhani Fard and A. Rahimi, A note on norm retrievable real Hilbert space frames, *J. Math. Anal. Appl.* 2021. (517)2, (2023) 126620.
2. R. Balan, P. G. Casazza and D. Edidin, On signal reconstruction without phase, *Appl. Comput. Harmonic Anal.* **20**(3), (2006) 345–356.
3. S. Botelho-Andrade, P. G. Casazza, D. Cheng, J. Haas, Tin T. Tran, J. C. Tremain, and Z. Xu, Phase retrieval by hyperplanes, arXiv:1703.02678v1, (2017).
4. J. Cahill, P. G. Casazza, J. Peterson and L. Woodland, Phase retrivial by projections, *Houston Journal of Mathematics*, **42**(2), (2016) 537–558. arXiv:1703.02657v1, (2017).
5. P. G. Casazza and D. Cheng, Associating vectors in $\mathbb{C}^n$ with rank 2 projections in $\mathbb{R}^{2n}$: with applications, arXiv:1703.02657v1, (2017).
6. P. G. Casazza, D. Ghoreishi, S. Jose and J. C. Tremain, Norm retrieval and phase retrieval by projections, *Axioms*, **6**, (2017) 1–15.
7. P. G. Casazza and G. Kutyniok, Frames and subspaces. In Wavelets, Frames, and Operator Theory. Contemporary Mathematics, *American Mathematical Society, Providence.* **345**, (2004) 87–113.
8. P. G. Casazza, G. Kutyniok and S. Li, Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.* **25**, (2008) 114–132,
9. O. Christensen, An introduction to frames and Riesz bases, (Birkhauser, Boston 2003).
10. M. A. Hasankhani Fard, Norm retrievable frames in $\mathbb{R}^n$, *Electronic Journal of Linear Algebra*, **31**, (2016) 425–432.