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CHARACTERIZATION OF n-JORDAN MULTIPLIERS THROUGH ZERO PRODUCTS

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Abstract. Let A be a unital C^*-algebra and X be a unital Banach A-bimodule. We characterize n-Jordan multipliers $T : A \longrightarrow X$ through the action on zero product. We also prove that each continuous linear mapping T from group algebra $L^1(G)$ into unital Banach A-bimodule X which satisfies a related condition is an n-Jordan multiplier.

1. Introduction

Let A be a Banach algebra and X be an A-bimodule. A linear map $T : A \longrightarrow X$ is called *left multiplier* [*right multiplier*] if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad [T(ab) = aT(b)],$$

and T is called a *multiplier* if it is both left and right multiplier. Also, T is called *left Jordan multiplier* [*right Jordan multiplier*] if for all $a \in A$,

$$T(a^2) = T(a)a, \quad [T(a^2) = aT(a)],$$

and T is called a *Jordan multiplier* if T is a left and a right Jordan multiplier.

It is clear that every left (right) multiplier is a left (right) Jordan multiplier, but the converse is not true in general, see for example [6].

A linear map D from Banach algebra A into an A-bimodule X is called *derivation* [*Jordan derivation*] if

$$D(ab) = D(a)b + aD(b), \quad [D(a^2) = D(a)a + aD(a)], \quad a, b \in A.$$

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Note that every derivation is a Jordan derivation, but the converse is fails in general [5]. It is proved by B. E. Johnson in [5, Theorem 6.3] that every Jordan derivation from C^*-algebra A into any A-bimodule X is a derivation.

Definition 1.1. Let A be a Banach algebra, X be a right A-module and let $T : A \longrightarrow X$ be a linear map. Then T is called *left n-Jordan multiplier* if for all $a \in A$, $T(a^n) = T(a^{n-1})a$. The *right n-Jordan multiplier* and *n-Jordan multiplier* can be defined analogously.

The two following results concerning characterization of n-Jordan multiplier presented by the first author in [6].

Theorem 1.2. [6, Theorem 2.3] *Let A be a unital Banach algebra and X be a unital Banach left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map such that*

$$a, b \in A, \quad ab = e_A \quad \Longrightarrow \quad T(ab) = aT(b). \tag{1.1}$$

Then T is a right n-Jordan multiplier.

Lemma 1.3. [6, Lemma 2.1] *Let A be a Banach algebra, X be a left A-module and let $T : A \longrightarrow X$ be a right Jordan multiplier. Then T is a right n-Jordan multiplier for each $n \geq 2$.*

Let A be a Banach algebra and X be a arbitrary Banach space. Then the continuous bilinear mapping $\phi : A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \quad a, b \in A. \tag{1.2}$$

Motivated by (1.2) the following concept was introduced in [1].

Definition 1.4. A Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (1.2) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

It is known that every C^*-algebra A and the group algebra $L^1(G)$ for a locally compact group G has the property $(\mathbb{B})$, [1].

Let $\mathfrak{J}(A)$ denote the subalgebra of A generated by all idempotents in A. If $A = \overline{\mathfrak{J}(A)}$, then we say that the Banach algebra A is generated by idempotents. Examples of such Banach algebras are given in [1].

Consider the following condition on a linear map T from Banach algebra A into a Banach A-bimodule X which is related to the condition (1.1).

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad aT(b) = 0. \tag{1.3}$$

A rather natural weakening of condition (1.3) is the following:

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad aT(b) + bT(a) = 0. \tag{1.4}$$

In this note, according to [7], we investigate whether those conditions characterizes n-Jordan multipliers.

2. Characterization of n-Jordan multipliers

Since all results which are true for left versions have obvious analogue statements for right versions, we will focus in the sequel just the right versions.

Theorem 2.1. *Let A be a unital C^*-algebra and X be a unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying (1.3). Then T is a right n-Jordan multiplier.*

We mention that Theorem 2.1 is also true for non-unital case, because every C^*-algebra A has a bounded approximate identity.

In view of Theorem 2.1, the next question can be raised. Dose Theorem 2.1 remain valid with condition (1.3) replaced by (1.4)?

Theorem 2.2. [2, Theorem 2.2] *Let A be a C^*-algebra and X be a Banach space and let $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \implies \phi(a, b) = 0, \qquad a, b \in A.$$

Then

$$\phi(ax, by) + \phi(ya, xb) = \phi(a, xby) + \phi(yax, b),$$

for all $a, b, x, y \in A$.

Our first main result is the following.

Theorem 2.3. *Let A be a unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4). Then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Corollary 2.4. *Let A be a commutative unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear mapping such that the condition (1.4) holds. Then T is an n-Jordan multiplier.*

Next we generalize Corollary 2.4 and give the affirmative answer to the preceding question.

Theorem 2.5. *Let A be a von Neumann algebra and X be a unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then T is a right n-Jordan multiplier.*

It is shown [3] that every C^*-algebra A is Arens regular and the second dual of each C^*-algebra is a von Neumann algebra. Hence by extending the continuous linear map $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and applying Theorem 2.5, we get the following result.

Corollary 2.6. *Suppose that A is a unital C^*-algebra and X is a unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then T is a right n-Jordan multiplier.*

Theorem 2.7. [4, Corollary 3.6] *Let A be Banach algebra, X be a Banach space and $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$a, b \in A, \quad ab = ba = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$. In particular, if A is generated by idempotents, then

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \qquad a, b \in A.$$

By using Theorem 2.7 we can obtain the following result.

Theorem 2.8. *Let A be a unital Banach algebra which is generated by idempotents and X be a symmetric unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Corollary 2.9. *Let A be a commutative unital Banach algebra such that $A = \overline{\mathfrak{J}(A)}$ and X be a symmetric unital left A-module. Let $T : A \longrightarrow X$ be a continuous linear map satisfying (1.4). Then T is an n-Jordan multiplier.*

Let $A = L^1(G)$ for a locally compact abelian group G. Then A is commutative and it is weakly amenable [3], but neither it is C^*-algebra nor generated by idempotents. Therefore Corollary 2.4 and Corollary 2.9 cannot be applied for it.

The following result shows that analogous of Corollary 2.4 is also true for group algebra.

Theorem 2.10. *Let $A = L^1(G)$ for a locally compact abelian group G. Suppose that X is a symmetric unital left A-module and $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4). Then T is an n-Jordan multiplier.*

References

1. J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, *Maps preserving zero products*, Studia Math., **193** (2009) 131-159.
2. J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, *Characterizing Jordan maps on C^*-algebras through zero products*, Proc. Roy. Soc. Edinb. Sect., **53** (2010) 543-555.
3. H. G. Dales, *Banach Algebras and Automatic Continuity*, LMS Monographs 24, Clarenden Press, Oxford, 2000.
4. H. Ghahramani, *On derivations and Jordan derivations through zero products*, Oper. Matrices, **8** (2014) 759-771.
5. B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Camb. Phil. Soc., **120** (1996) 455-473.
6. A. Zivari-Kazempour, *Characterization of n-Jordan multipliers*, Vietnam J. Math., **50** (2022) 87-94.
7. A. Zivari-Kazempour and M. Valaei, *Characterization of n-Jordan multipliers through zero products*, J. Analysis, **30** (2022) 1059-1067.