The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

CONVOLUTION, ERROR FUNCTION AND A NEW SPECIAL FUNCTION IN CLASS OF UNIVALENT ANALYTIC FUNCTION

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Abstract. The main objective of this paper is to introduce a new special class of analytic univalent functions based on a combination of the Error function and a new function, that we create with the help of convolution. we examine several properties of this class, such as, Weighted mean, Coefficient estimate and extreme points.

1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $\mathcal{A}$ the well-known class of analytic and normalized functions of in $\mathbb{U}$. we note that each function f in $\mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}, a_n \in \mathbb{C}). \tag{1.1}$$

we say that a function f is univalent in $\mathbb{U}$ if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathbb{U}$ with $z_1 \neq z_2$. The family of all univalent functions f in $\mathbb{U}$ is denoted by $\mathcal{S}$ [2, 4]. The subclass of $\mathcal{A}$ create withe changing negative coefficients and are of the type

2010 *Mathematics Subject Classification. 30C45, 30C50*

Key words and phrases. Error function, Convolution, Univalent function, Weighted mean, Coefficient estimate.

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.2}$$

The convolution or Hadamard product $f(z)$ and $g(z)$ for f to form (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ for more details see [2].
The Error Function and Subclasses of Analytic Univalent Functions introduce by Sayedain and Najafzadeh [5], is form

$$\begin{aligned} \mathrm{E_r f(z)} &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n, \quad (z \in \mathbb{C}). \end{aligned} \tag{1.3}$$

Let $h(z) = z + (\frac{3}{e} - 2)z^n$, $(n = 0, 1, 2, ...)$ and the Taylor series of this h

$$h(z) = z - \sum_{n=2}^{\infty} \frac{(-1)^n (2n-1)}{n!} z^n, \tag{1.4}$$

Definition 1.1. The function $H(z)$ denote by convolution $h(z)$ and $\mathrm{E_r f(z)}$

$$\begin{aligned} H(z) &= h(z) * (2z - \mathrm{E_r(f)}) * \mathrm{f(z)} \\ &= z - \sum_{n=2}^{\infty} \frac{1}{n((n-1)!)^2} a_n z^n. \end{aligned} \tag{1.5}$$

Where $f(z)$ is (1.2). new, a function of the form (1.2) is in the class $\mathcal{W}_Q(a, b)$ if it satisfies the condition

$$\mathrm{Re} \left\{ \frac{H(z) + zH'(z) + az^2 H''(z) - z}{azH'(z) + (1-b)H(z)} \right\} > Q, \quad {\scriptstyle (0 \leq Q < 1)}. \tag{1.6}$$

Where $0 \leq a, b \leq 1$, $a < b$ and $H(z)$ are give by (1.5), also $H'(z), H''(z)$ are first and second order derivatives, respectively [3].

2. MAIN RESULTS

In the folloing theorem, we express a condition for the functions that belong to the class $\mathcal{W}_Q(a, b)$.

Theorem 2.1. *Let $f(z)$ of the form*(1.2)*. f belong to the class $\mathcal{W}_Q(a, b)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\frac{n+1}{n} + a(n(Q+1)) + Q(1-b)}{((n-1)!)^2} a_n \leq 1 + Q(b - a - 1), \quad {\scriptstyle (0 \leq Q < 1)}. \tag{2.1}$$

Proof. By calculating the derivatives of the first and second order and placing in (1.5) and also consider "z" as a real number that is moved to $z \to 1^-$ we have

$$\frac{1 - \sum_{n=2}^{\infty} \frac{a(n-1) + \frac{1}{n} + 1}{((n-1)!)^2} a_n}{(1+a-b) - \sum_{n=2}^{\infty} \frac{an-b+1}{((n-1)!)^2} a_n} > Q$$

$$Q(b-a-1) + 1 - \sum_{n=2}^{\infty} \frac{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}{((n-1)!)^2} a_n \geq 0$$

So, we proved that if $f(z)$ belong to the class $\mathcal{W}_Q(a,b)$ the related (2.1) holds. Conversely, it is easily proven. see [5]. □

The results is sharp for example $g(z) = z - \frac{1+Q(b-a-1)}{Q(2a-b+1)+2a+1.5} z^2$.

Theorem 2.2. *$\mathcal{W}_Q(a,b)$ is a convex set.*

Proof. It is enough to show for $f_i(z)$ belong to the class $\mathcal{W}_Q(a,b)$, then $\sum_{i=1}^{m} \lambda_i f_i(z) \in \mathcal{W}_Q(a,b)$ where $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda_i \geq 0$. □

3. ON GEOMETRIC PROPERTIES

In the following theorem, we introduce the functions that belong to the class $\mathcal{W}_Q(a,b)$ and are the extreme points of the set. we will show that they have such a property [1].

Theorem 3.1. *Let $f_n(z) = z - \frac{((n-1)!)^2(1+Q(b-a-1))}{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1} z^n$, $(n = 2, 3, ...)$ also $f_1(z) = z$. then $\sum_{n=1}^{\infty} \mu_n f_n(z) \in \mathcal{W}_Q(a,b)$ if and only if $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n \geq 0$.*

Proof. Let assume first $\sum_{n=1}^{\infty} \mu_n f_n(z) \in \mathcal{W}_Q(a,b)$ for $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n \geq 0$. we will show that $f_n \in \mathcal{W}_Q(a,b)$ $(n = 2, 3, ...)$. Refer theorem(2.1)

$$a_n \leq \frac{1+Q(b-a-1)((n-1)!)^2}{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}$$

therefore by letting

$$\mu_n = \frac{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}{1+Q(b-a-1)((n-1)!)^2} a_n$$

and that $\mu_1 = 1-(\mu_2+\mu_3+...)$ we conclude the required result. Conversely is easily. $\square$

Theorem 3.2. *Let $f_1(z) = z - \sum_{n=2}^{\infty} a_{n,1}z^n$ and $f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2}z^n$ belongs to the class $\mathcal{W}_Q(a,b)$, then the weighted mean of f_1, f_2 in to $\mathcal{W}_Q(a,b)$.*

Proof. Let $H_t(z) = \frac{1}{2}(1+t)f_1(z) + \frac{1}{2}(1-t)f_2(z)$ we have

$$H_t(z) = z - \frac{1}{2}\sum_{n=2}^{\infty}((1+t)a_{n,1} + (1-t)a_{n,2})z^n$$

since f_1 and f_2 are in the class $\mathcal{W}_Q(a,b)$, so by theorem(2.1) we have

$$\sum_{n=2}^{\infty} \frac{a(n(Q+1)) + Q(1-b) + \frac{n+1}{n}}{((n-1)!)^2}[\frac{1}{2}(1+t)a_{n,1} + \frac{1}{2}(1-t)a_{n,2}]$$
$$\leq \frac{1}{2}(1+t)(1+Q(b-a-1)) + \frac{1}{2}(1-t)(1+Q(b-a-1))$$

Which the above expression is equal to $Q(b-a-1)+1$. So the condition of theorem(2.1) for $H_t(z)$ is established and therefore the proof is complete. $\square$

References

1. S. Altınkaya and S.O. Olatunji, *Generalized distribution for analytic function classes associated with error functions and Bell numbers*, Bol. Soc. Mat. Mex, **26** (2020) 377–384.
2. P. L. Duren, *Univalent functions, Grundelheren der Mathematischen Wissenchaften* Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
3. Sh. Najafzadeh, *Some results on univalent holomorphic functions based on q-analogue of Noor operator*, Int. J. Appl. Math., **32(5)** (2019) 775-784.
4. C. Ramachandran, L. Vanitha and S. Kanas, *Certain results on q-starlike and q-convex error functions*, Math. Slovaca., **68** (2018) 361-368.
5. S. H. Sayedain Boroujeni, Sh. Najafzadeh, *Error Function and Certain Subclasses of Analytic Univalent Functions*, Sahand Commun. Math. Anal., **20(1)** (2023) 107-117.

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