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ON THE POSITIVE OPERATORS AND LINEAR MAPS

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Abstract. In this work, we first prove an numerical inequality. Then, we present some inequalities for positive operators and linear maps.

1. Introduction

In what follows, we denote by $\mathbb{B}(\mathbf{H}))$ the space of all bounded linear operators on a Hilbert space $(H, <.,.>)$. We say the operator A is positive if $< Ax, x >\geq 0$ for all $x \in \mathbf{H}$ and write $A \geq 0$, is invertible positive if $< Ax, x >> 0$ for all $x \in \mathbf{H}$ and write $A > 0$. For two selfadjoint operators $A, B \in \mathbb{B}(\mathbf{H})$, we say $A \geq B(A > B)$ if $A - B \geq 0$ $(A - B > 0)$, respectively. The adjoint of the operator A define by A^* and its absolute value by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$. A linear map Φ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B \in \mathbb{B}(\mathbf{H})$ such that A and B are invertible positive and $0 \leq \nu \leq 1$, we utilize the following notations to define the geometric mean $A\sharp_\nu B$ and the arithmetic mean $A\nabla_\nu B$, respectively,

$$A\#_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}} \quad \textit{and} \quad A \bigtriangledown_\nu B = \nu A + (1-\nu)B.$$

For $A, B \in \mathbb{B}(\mathbf{H})$ such that are invertible positive and $0 \leq \nu \leq 1$, we have operator Young inequality as follows:

$$A\#_\nu B \leq A\nabla_\nu B. \tag{1.1}$$

The Lowner-Heinz theorem [6] states that if $A, B \in \mathbb{B}(\mathbf{H})$ are positive, then for $0 \leq p \leq 1$,

$$A \leq B \qquad \text{implies} \quad A^p \leq B^p. \tag{1.2}$$

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In general (1.2) is not true for $p > 1$. M. Lin [5] reversed (1.1) using the Katorovich constant as follows: If $0 < m \leq A, B \leq M$ and Φ are a unital positive linear map. Then

$$\varphi^2(A\nabla B) \leq K^2(h)\varphi^2(A\#B), \varphi^2(A\nabla B) \leq K^2(h)(\varphi(A)\#\varphi(B))^2, \tag{1.3}$$

where $K(h) = \frac{(1+h)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

2. Main results

Furuichi et al. in [4] showed that for $0 < x \leq 1$ and $0 \leq \nu \leq 1$, the following inequality holds:

$$m_\nu(x)x^\nu \leq (1-\nu) + \nu x, \tag{2.1}$$

where $m_\nu(x) = 1 + \frac{2^\nu \nu(1-\nu)(x-1)^2}{(x+1)^{1+\nu}}$ and $1 \leq m_\nu(x)$. The next Lemma is an refinement of (2.1).

Lemma 2.1. *Let $0 < x \leq 1$. If $0 \leq \nu \leq \frac{1}{2}$, then*

$$m_{2\nu}(\sqrt{x})x^\nu + \nu(\sqrt{x}-1)^2 \leq (1-\nu) + \nu x, \tag{2.2}$$

where $m_{2\nu}(\sqrt{x})$ defined as (2.1).

Proof. Letting $0 \leq \nu \leq \frac{1}{2}$. By an simple computation

$$(1-\nu) + \nu x - \nu(\sqrt{x}-1)^2 = 2\nu\sqrt{x} + (1-2\nu).$$

By applying (2.1) for the relation above,

$$m_{2\nu}(\sqrt{x})x^\nu \leq (1-2\nu) + 2\nu\sqrt{x}.$$

Therefore, (2.2) is proved. □

For a operator version of the inequality (2.2), see Theorem 2.2.

Theorem 2.2. *Let $A, B \in \mathbb{B}(H)$ are two invertible positive operators such that $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$ for some positive real numbers m, m', M, M'. Then for $0 \leq \nu \leq \frac{1}{2}$*

$$A\nabla_\nu B \geq m_{2\nu}(\sqrt{h})A\sharp_\nu B + \nu(A\nabla B - A\sharp B), \tag{2.3}$$

where $m_{2\nu}(\sqrt{x})$ defined as (2.1).

Proof. The condition $0 < m \leq A \leq m' < M' \leq B \leq M$ ensure us that $Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \subset [h', h]$, where $h = \frac{M}{m}$ and $h = \frac{M'}{m'}$. On the other hand, (2.2) implies that

$$\min_{\sqrt{h'} \leq \sqrt{x} \leq \sqrt{h} \leq 1} m_{2\nu}(\sqrt{x})x^\nu + \nu(\sqrt{x}-1)^2 \leq (1-\nu) + \nu x, \tag{2.4}$$

If we set $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the inequality (2.4) and use from the decreasing property $m_{2\nu}(\sqrt{x})$, the following inequality deduces

$$\begin{aligned} &\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\nu)I \\ &\geq m_{2\nu}(\sqrt{h})\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu + \nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}). \end{aligned} \tag{2.5}$$

Multiplying both sides of the inequality (2.5) by $A^{\frac{1}{2}}$, we can get to the desired result. Similarly, the condition $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ concludes the desired result. □

Remark 2.3. As $A\nabla B - A\sharp B \geq 0$. By (2.3), we have $A\nabla_{\nu} B \geq m_{2\nu}(\sqrt{h})A\sharp_{\nu} B$.

2.1. **The higher powers using positive mps.**

Lemma 2.4. [3] *Let $A \in \mathbb{B}(H)$ be positive and Φ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \leq \Phi\left(A^{-1}\right). \tag{2.6}$$

Lemma 2.5. [2]-[1] *Let $A, B \geq 0$. Then for $1 \leq r < \infty$*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.7}$$

$$\|A^r + B^r\| \leq \|(A+B)^r\|. \tag{2.8}$$

Theorem 2.6. *Let $A, B \in \mathbb{B}(H)$ are two invertible positive operators such that $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$ or $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ for some positive real numbers $m, m^{'}, M, M^{'}$ and Φ be a normalized positive linear map. Then for every $0 \leq \nu \leq \frac{1}{2}$*

$$\Phi^2(A\nabla_{\nu} B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^2 \Phi^2(A\sharp_{\nu} B), \tag{2.9}$$

$$\Phi^2(A\nabla_{\nu} B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^2 (\Phi(A)\sharp_{\nu}\Phi(B))^p, \tag{2.10}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant and $m_{2\nu}(\sqrt{h})$ is as defined in (2.1).

Proof. It is trivial that$A + MmA^{-1} \leq M + m$and$B + MmB^{-1} \leq M + m$. An simple computation show that

$$\Phi\left(A\nabla_v B\right) + Mm\Phi\Big(A^{-1}\nabla_v B^{-1}\Big) \leq M + m. \tag{2.11}$$

By applying (2.7), (2.6), (2.3)and(2.11), respectively, one can check that

$$\begin{aligned}
&\left\|\Phi(A\nabla_{\nu} B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right))Mmm_{2\nu}(\sqrt{h})\Phi^{-1}(A\sharp_{\nu} B)\right\| \\
&\leq \frac{1}{4}\left\|\Phi(A\nabla_{\nu} B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) + Mmm_{2\nu}(\sqrt{h})\Phi^{-1}(A\sharp_{\nu} B)\right\|^2 \\
&\leq \frac{1}{4}\left\|\Phi(A\nabla_{\nu} B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) + Mmm_{2\nu}(\sqrt{h})\Phi(A^{-1}\sharp_{\nu} B^{-1})\right\|^2 \\
&= \frac{1}{4}\left\|\Phi(A\nabla_{\nu} B) + Mm\Phi\left(\nu\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) + m_{2\nu}(\sqrt{h})(A^{-1}\sharp_{\nu} B^{-1})\right)\right\|^2 \\
&\leq \frac{1}{4}\left\|\Phi(A\nabla_{\nu} B) + Mm\Phi(A^{-1}\nabla_{\nu} B^{-1})\right\|^2 \\
&\leq \frac{(M+m)^2}{4}.
\end{aligned}$$

This proves the inequality (2.9). The inequality (2.10) can prove similarly. □

Corollary 2.7. *Let $A, B \in \mathbb{B}(H)$ are two invertible positive operators such that $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$ or $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ for some positive real numbers $m, m^{'}, M, M^{'}$ and Φ be a normalized positive linear map. Then for $p > 0$ and every $0 \leq \nu \leq \frac{1}{2}$*

$$\Phi^p(A\nabla_\nu B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^p \Phi^p(A\sharp_\nu B), \quad (2.12)$$

$$\Phi^p(A\nabla_\nu B + \nu Mm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^p (\Phi(A)\sharp_\nu \Phi(B))^p, \quad (2.13)$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant and $m_{2\nu}(\sqrt{h})$ is as defined in (2.1).

Proof. If $0 < p \leq 2$, then $0 < \frac{p}{2} \leq 1$. Thus, by Theorem 2.6, we obtain the desired results. Letting $p > 2$. By (2.8) and the same method as used in Theorem 2.6 the inequalities above conclude. □

Remark 2.8. It is clear that $A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \geq 0$. Thus,

$$\Phi^p(A\nabla_\nu B) + (\nu Mm)^p\Phi^p(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \geq \Phi^p(A\nabla_\nu B).$$

In result,

$$\begin{aligned}\|\Phi^p(A\nabla_\nu B)\| &\leq \|\Phi^p(A\nabla_\nu B) + (\nu Mm)^p\Phi^p(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\| \\ &\leq \|(\Phi(A\nabla_\nu B) + (\nu Mm)\Phi(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))^p\| (by (2.8)).\end{aligned}$$

On the other hand, by (2.1)., $m_{2\nu}(\sqrt{h}) \geq 1$. This shows that hand-left side of (2.12) and (2.13) is a norm refinement of (1.3) and hand-right side of (2.12) and (2.13) are tighter than (1.3).

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