The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

MAJORIZATION AND STOCHASTIC LINEAR MAPS IN VON NEUMANN ALGEBRAS

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Abstract. The aim of the present paper is to introduce semi-doubly stochastic and (weak)majorization on a non commutative measure space $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a semi finite von Neumann algebra with a normal faithful trace τ.

1. Introduction

Since Hardy, Litttewood, and Pólya in 1929 introduced the concept of majorization, many mathematicians have discussed the weak majorization and manjorization in various circumstances with several applications. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. x is said to be majorized and denoted by y $x \prec y$, if $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$, for all $1 \leq k \leq n$ and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$,where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ and $y^{\downarrow} = (y_1^{\downarrow}, \ldots, y_n^{\downarrow})$ are obtained from x and y by rearranging their components in decreasing order. Moreover, the study of (weak)majorization has been successful in the theory of matrices via comparison of eigenvalues by Ando in 1982. On the other hand, the doubly stochastic matrices and maps have been studied in connection with majorization theory by Mirsky, Chong, Alberti and Uhlman.

1991 *Mathematics Subject Classification.* Primary 47B60; Secondary 60E15.
Key words and phrases. majorization, singular values, stochastic maps.

Definition 1.1. An $n \times n$ matrix $D = (a_{ij})$ is called doubly stochastic if $D\mathbf{1} = \mathbf{1}$ and $D^*\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and D^* is the adjoint matrix of D.

Theorem 1.2. [5] For $X, Y \in \mathbb{R}^n$, the following statements are equivalent:

(1) $X \prec Y$.
(2) X is in the convex hull of all vectors obtained by permuting the coordinates of Y.
(3) $X = DY$ for some doubly stochastic matrix D.

Definition 1.3. Let A and B are two $m \times n$ matrices. A is majorized by B in symbols $A \prec B$ if there is a doubly stochastic $m \times m$ matrix D such that $A = DB$.

The theory of (weak)majorization has been developed for real- valued measurable functions on abstract measure space based on the theory rearrangements by Chong and Sakai. In the case of a σ-finite measure space $(X, \mathcal{A}, \mu)$, the notion of decreasing rearrangement can be defined for non-negative integrable functions. For a finite measure space $(X, \mathcal{A}, \mu)$, Ryff considered the class of all linear operators $T : L^1(X, \mu) \to L^1(X, \mu)$ for which $Tf \prec f$, for all $f \in L^1(X, \mu)$. This class is denoted by $\mathcal{DS}(L^1(X, \mu))$ and each element of this class is called doubly stochastic operators. For σ-finite measure space $(X, \mathcal{A}, \mu)$, in [1] the semi-doubly stochastic operator is introduced and the set of all these operators is denoted by $\mathcal{SDS}\big(L^1(X, \mu)\big)$. For a non-negative $f \in L^1(X, \mu)$, let $\mathcal{S}_f := \{Sf; S \in \mathcal{SDS}(L^1)\}$ and $\Omega_f := \{h \in L^1; h \geq 0 \text{ and } h \prec f\}$. It is easily seen that both sets $\mathcal{S}_f$ and Ω_f are convex subsets of L^1. It has been proved that $\mathcal{S}_f$ is dense in Ω_f [1].

2. Main results

In this section we study the relation between majorization and doubly stochastic maps on a semi finite von Neumann algebras. Throughout this section $\mathcal{M}$ is a semi finite von Neumann algebra on a Hilbert space $\mathcal{H}$ and τ is a faithful normal semi finite trace on $\mathcal{M}$. We fix a couple $(\mathcal{M}, \tau)$ as a non-commutative measure space. For positive operator x affiliated with We fix a couple $(\mathcal{M}, \tau)$, $e_I(x)$ will denote the spectral projection of x corresponding to an interval I in $[0, \infty)$. A closed and densely defined linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is said to be τ-measurable if x affiliated with $\mathcal{M}$, and there exists $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. The collection of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a complex $*$-algebra with unit element 1. The von Neumann algebra $\mathcal{M}$ is a $*$-subalgebra of $L_0(\mathcal{M})$. For each $\mathfrak{L}$ of $L_0(\mathcal{M})$, the set of all positive elements in $\mathfrak{L}$ is denoted by $\mathfrak{L}_+$. The closure of $L_1(\mathcal{M})$ in $L_0(\mathcal{M})$ is denoted by $\widetilde{\mathtt{G}}$.

Let $x \in L_0(\mathcal{M})$ and $t > 0$. The t-th singular value of x (or generalized s-numbers) is the number denoted by $\mu_t(x)$ and for each $t \in \mathbb{R}_0^+$ is defined

by
$$\mu_t(x) = \inf\{\| xe \|: e \in \mathcal{P}(\mathcal{M}), \tau(1-e) \leq t\}.$$
For $0 < p < \infty$, $L_p(\mathcal{M}, \tau)$ is defined as the set of all τ-measurable operators x such that
$$\| x \|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty. \tag{2.1}$$
Moreover, we put $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\| \cdot \|_\infty$ the usual operator norm. For simplicity from now on $L_p(\mathcal{M}, \tau)$ will denoted by $L_p(\mathcal{M})$. Let $1 \leq p < \infty$, an operator $x \in \mathcal{M}$ is said to be locally integrable if there exists $\delta > 0$ such that
$$\int_0^\delta \mu_t(x)^p \, dt < \infty.$$
The set containing all these operators is denoted by $\mathfrak{L}^p_{loc}(\mathcal{M})$. Note that in particular, all bounded operators $a \in \mathcal{M}$ are of this class. Moreover,
$$\int_0^\delta \mu_t(x)^p \, dt \geq \mu_\delta(x)^{p-1} \int_0^\delta \mu_t(x) \, dt$$
implies that $\mathfrak{L}^p_{loc}(\mathcal{M}) \subset \mathfrak{L}^1_{loc}(\mathcal{M})$ for each $p \geq 1$ [3].

Definition 2.1. Let a, b be positive τ-measureable operators. We say that a is submajorized (weakly majorized) by b in symbol $a \prec_w b$, if $\int_0^s \mu_t(a)dt \leq \int_0^s \mu_t(b)dt$ for all $s > 0$. Moreover a is said to be majorized by b and is indicated by $a \prec b$, if $a \prec_w b$ and $\int_0^\infty \mu_t(a)dt = \int_0^\infty \mu_t(b)dt$.

Let φ be a linear map from $\mathcal{M}$ to itself. φ is positive if $\varphi(a)$ is positive for every $a \in \mathcal{M}_+$, φ is unital if $\varphi(1) = 1$ and φ is trace preserving if $\tau(\varphi(a)) = \tau(a)$.

Definition 2.2. [2] A positive linear map $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ is called doubly stochastic if it is unital and trace preserving. φ is called doubly substochastic if $\varphi(1) \leq 1$ and $\tau(\varphi(a)) \leq \tau(a)$ for all $a \in \mathcal{M}_+$. The set of all doubly stochastic (resp. doubly substochastic) linear maps on $\mathcal{M}$ is denoted by $\mathcal{DS}(\mathcal{M})$(resp. $\mathcal{DSS}(\mathcal{M})$).

In the following two propositions, which are proved in [2], the relations between (weak)majorization and doubly (sub)stochastic maps are investigated.

Proposition 2.3. Let $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ be a positive linear map. Then

(1) $\varphi(a) \prec_w a$ for all $a \in \mathcal{M}_+$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$.
(2) $\varphi(a) \prec a$ for all $a \in \mathcal{M}_+$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$ and φ is trace preserving (hence $\varphi \in \mathcal{DS}(\mathcal{M})$ when $\tau(1) < \infty$).

Proposition 2.4. Let $a, b \in L_0(\mathcal{M})$.

(1) If $\tau(1) < \infty$ and $b \in L_1(\mathcal{M})$, then $a \prec b$ if and only if there exists $\varphi \in \mathcal{DS}(\mathcal{M})$ such that $a = \varphi(b)$.
(2) If $b \in L_p(\mathcal{M})$ with $1 \leq p < \infty$, or if $b \in \mathcal{M}$ and $a \in \widetilde{\mathtt{G}}$, then $a \prec_w b$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$ such that $a = \varphi(b)$.

Moreover, a normal and completly positive φ can be chosen in each (1) and (2) if $b \in L_p(\mathcal{M})$, $1 \leq p < \infty$, or $a \in \mathtt{G} = \widetilde{\mathtt{G}} \cap \mathcal{M}$.

Theorem 2.1. [4] *Let x, y be operators in $L_0(\mathcal{M})$. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$\frac{1}{r} \mid xy^* \mid^r \prec_w \frac{1}{p} \mid x \mid^p + \frac{1}{q} \mid y \mid^q \ . \tag{2.2}$$

Moreover, if $x, y \in L_1(\mathcal{M})$ are bounded operators or $xy \in \mathfrak{L}^2_{loc}(\mathcal{M})$, then there exists a $\varphi \in \mathcal{DS}(\mathcal{M})$ such that $a = \varphi(b)$.

$$\frac{1}{r} \mid xy^* \mid^r \prec \frac{1}{p} \mid x \mid^p + \frac{1}{q} \mid y \mid^q, \tag{2.3}$$

if and only if $\mid x \mid^p = \mid y \mid^q$.

Corollary 2.5. Let $x, y \in L_1(\mathcal{M})$ are bounded operators. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, there exists $\varphi \in \mathcal{DSS}(\mathcal{M})$ such that

$$\mid xy^* \mid^r = \varphi\left(\frac{r}{p} \mid x \mid^p + \frac{r}{q} \mid y \mid^q\right).$$

Moreover, if $\tau(1) < \infty$, then $\varphi \in \mathcal{DS}(\mathcal{M})$

3. Conclusion

For $a \in L_0(\mathcal{M})_+$, let $\mathcal{S}_a := \{\varphi(x);\ \varphi \in \mathcal{DSS}(\mathcal{M})\}$, $\mathcal{D}_a := \{\varphi(x);\ \varphi \in \mathcal{DS}(\mathcal{M})\}$ and $\Omega_a := \{b \in L_0(\mathcal{M})_+;\ b \prec a\}$. Sets $\mathcal{S}_a$, $\mathcal{D}_a$ and Ω_a are convex. Proposition 2.4 (part (1)) implies that If $\tau(1) < \infty$ and $a \in L_1(\mathcal{M})$, then $\mathcal{S}_a = \mathcal{D}_a$. When $\tau(1) = \infty$, it is not clear for us whether or not $\mathcal{S}_a = \Omega_a$. If we consider $(\Omega\mathcal{S})_a := \{b \in L_0(\mathcal{M})_+;\ b \prec_w a\}$, then Proposition 2.4 (part (2)) implies that $\mathcal{S}_a = (\Omega\mathcal{S})_a$ when $a \in L_1(\mathcal{M})$.

Acknowledgement

This work was supported by the Department of Mathematical Sciences at Isfahan University of Technology, Iran.

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