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GRADIENT ESTIMATE FOR $\Delta u + au(\log u)^p + bu = f$ UNDER THE RICCI SOLITON CONDITION

SAKINEH HAJIAGHASI, SHAHROUD AZAMI

Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran
s.hajiaghasi@edu.ikiu.ac.ir; azami@sci.ikiu.ac.ir

Abstract. In this paper, we consider the gradient estimate of the following equation
$$\Delta u + au(\log u)^p + bu = f,$$
for some smooth function f and real constants a, b and we obtain an upper bound for gradient of u. As an application we obtain the gradient bound for the Riemannian manifold with Bakry-Émery Ricci curvature.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold with a fixed base point $O \in M$. Consider the following lower bound on the Ricci curvature
$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \tag{1.1}$$
for some constant $\lambda \geq 0$ and smooth vector field X which satisfies the following condition:
$$|X|(y) \leq \frac{K}{d(y, O)^\alpha}, \qquad \forall y \in M. \tag{1.2}$$
Here $d(y, O)$ reperesent the distance from O to y, K is a positive real constant, and $0 \leq \alpha < 1$.
In the pioneering work of Zhang and Zhu [3], they proposed following main

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conditions (1.1) and (1.2) on a Riemannian manifold and moreover the following volume noncollapsing condition

$$Vol(B(x,r)) \geq \rho, \tag{1.3}$$

for all $x \in M$ and some constant $\rho > 0$. Based on this assumptions they obtained Volume comparison theorem, Isoperimetric inequality and Sobolev inequality which leaded to the Elliptic and Parabolic gradient estimates.
Gradient estimate for the solutions of the Poisson equation and heat equation are very powerful tools in geometry and analysis. As an important application Li and Yau [1] deduced a Harnack inequality and also they obtained upper and lower bounds for heat kernel under the Dirichlet and Neumann boundary conditions. Recently Peng .et.al [2], stablished Yau-type gradient estimates for following equation on Riemannian manifolds

$$\Delta u + au(\log u)^p + bu = 0,$$

where $a, b \in \mathbb{R}$ and p is a rational number with $p = \dfrac{k_1}{2k_2+1} \geq 2$, where k_1 and k_2 are positive integer numbers.
In this paper, using the sufficient instrument like Sobolev inequality and Volume comparison Theorem from [3] and with the same method we want to obtain gradient estimate for the smooth function u which satisfies

$$\Delta u + au(\log u)^p + bu = f, \tag{1.4}$$

here a, b and $p > 0$ are real constant and $f : M \to \mathbb{R}$ is a smooth function.

2. MAIN RESULTS

We may use following isoperimetric and sobolev inequality. The proof process is just like [?], we can prove the theorem for any $r \leq r_0 = r_0(n, K_1, K, \alpha, \rho)$.

Theorem 2.1 (Isoperimetric inequality)**.** *Let M be a Riemannian manifold equiped with the Ricci soliton which next three conditions hold on it.*

$$Ric + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \qquad |X|(y) \leq \frac{K}{d(y,O)^\alpha}, \;\; Vol(B(x,1)) \geq \rho,$$

for all $x \in M$ and some constant $\rho > 0$ and $K \geq 0$. (we could just have the first two equations when $\alpha = 0$). Then there is a constant $r_0 = r_0(n, K_1, K, \alpha, \rho)$ such that for any $r \leq r_0$ and $f \in C_0^\infty(B(x,r))$, we have

$$\mathrm{ID}_n^*(B(x,r)) \leq C(n)r.$$

Here $\mathrm{ID}_n^*(B(x,r))$ is the isoperimetric constant defined by

$$\mathrm{ID}_n^*(B(x,r)) = Vol(B(x,r))^{\frac{1}{n}} . \sup_\Omega \left\{ \frac{Vol(\Omega)^{\frac{n-1}{n}}}{Vol(\partial\Omega)} \right\},$$

where the supremum is taken over all smooth domains $\Omega \subset B(x,r)$ with $\partial\Omega \cap \partial B(x,r) = \emptyset$.

Theorem 2.2 (Sobolev inequality)**.** *Under the same conditions as in the above theorem, we have the following Sobolev inequalities.*

$$\left(\oint_{B(x,r)}|f|^{\frac{n}{n-1}}dg\right)^{\frac{n-1}{n}}\leq C(n)r\oint_{B(x,r)}|\nabla f|dg,$$

and

$$\left(\oint_{B(x,r)}|f|^{\frac{2n}{n-2}}dg\right)^{\frac{n-2}{n}}\leq C(n)r^2\oint_{B(x,r)}|\nabla f|^2dg.$$

Moreover, for the case that $X=\nabla f$ for some smooth function f, we get

$$\left(\oint_{B(x,r)}|f|^{\frac{n}{n-1}}dg\right)^{\frac{n-1}{n}}\leq C(n)r\oint_{B(x,r)}|\nabla f|dg.$$

Theorem 2.3 (Volume comparison)**.** *Assume that for an n-dimension Riemannian manifold, (1.1) and (1.2) hold. Suppose in addition that the volume non-collapsing condition holds*

$$Vol(B(x,1))\geq\rho,$$

for positive constants $\rho>0$, $K\geq 0$ and $0\leq\alpha<1$, then for any $0<r_1<r_2\leq 1$, we have the volume ratio bound as follows

$$\frac{Vol(B(x,r_2))}{r_2^n}\leq e^{C(n,K_1,K,\alpha,\rho)[K_1(r_2^2-r_1^2)+K(r_2-r_1)^{1-\alpha}]}.\frac{Vol(B(x,r_1))}{r_1^n}. \quad (2.1)$$

In particular, this result are true by considering the gradient soliton vector field $V=\nabla f$.

Here is our main result:

Theorem 2.4. *Suppose that on a Riemannian manifold M^n, (1.1), (1.2) and (1.3) hold. For $q>\frac{n}{2}$, if u and f be smooth functions such that (1.4) holds with $0\leq u\leq l_1$ and $|(\log u)^p|\leq l_2$ for constants l_1,l_2, then there exists a positive constant $r_0=r_0(n,N,K,\alpha,\rho,l_1,l_2,l_3)$ such that for any $x\in M$ and $0<r\leq r_0$ we have*

$$\sup_{B(x,\frac{1}{2}r)}|\nabla u|^2\leq C(n,\lambda,K,\alpha,\rho,l_1,l_2)\left[(\|f\|^*_{2q,B(x,r)})^2+r^{-2}(\|u\|^*_{2,B(x,r)})^2\right].$$

As an application we conclude:

Corollary 2.5. *Suppose that the following condition holds for a gradient Ricci soliton*

$$\mathrm{Ric}+Hessh\geq-\lambda g,$$

and more over we have two condition for potential function h as follows

$$|h(y)-h(z)|\leq K_1d(y,z)^\alpha,\ \ and \quad \sup_{x\in M,0\leq r\leq 1}\left(r^\beta\|\nabla h\|^*_{q,B(x,r)}\right)\leq K_2.$$

Then there is a constant $r_0 = r_0(n, \lambda, K_1, K_2, \alpha, \beta, l_1, l_2)$*, such that by the same conditions as last theorem, the solution of (1.4) satisfies*

$$\sup_{B(x,\frac{r}{2})} |\nabla u|^2 \leq C(n, \lambda, K_1, K_2, \alpha, \beta, l_1, l_2)\left[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|h\|^*_{2q,B(x,r)})^2\right],$$

for any $q > \dfrac{n}{2}$.

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