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THE LOWER BOUND OF THE FIRST EIGENVALUES FOR A QUASILINEAR OPERATOR

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Abstract. In this paper, we use a special smooth function $f : \Omega \to \mathbb{R}$ on a bounded domain of a Riemannian manifold to estimate the lower bound of the first eigenvalue for quasilinear operator $Lf = -\Delta_p f + V|f|^{p-2}f$.

1. Introduction

It is well known that studying the eigenvalues and eigenfunctions of the Laplacian play an important role in global differential geometry since they reveal important relations between geometry of the manifold and analysis. So far, there have been some progress on the geometric operator as bi-Laplace, p-Laplace, and (p,q)-Laplace associated to a Riemannian metric g on a compact Riemannian manifold M^n. For instance, Lichnerowicz-type estimate had been studied in some research papers for the p-Laplace [4], p-Laplace with integral curvature condition [5], and recently investigated for the first eigenvalue of buckling and clamped plate problems in [3].
In this paper, we are going to study the first eigenvalue of following quasilinear operator which was introduced in [1], studied under considering different bounded Ricci curvature.
Let (M^n, g, dv) be a compact Riemannian manifold with volume element dv,

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the quasilinear operator on M defines as

$$Lf = -\Delta_p f + V|f|^{p-2}f. \tag{1.1}$$

Here V is a nonnegative smooth function on M, and for $p \in (1, \infty)$ the p-Laplace operator is defined as

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f).$$

Corresponding to the p-Laplacian we have the following eigenvalue equation

$$\begin{cases} Lf = \mu|f|^{p-2}f & on \quad M \\ f = 0 \ \ (Dirichlet) \ \ or \ \ \dfrac{\partial f}{\partial \nu} = 0 \ \ (Neumann) & on \quad \partial M \end{cases}$$

where ν is the outward normal on ∂M. The first nontrivial Dirichlet eigenvalue for M is given by

$$\mu_{1,p}(M) = \inf_{f \in W_0^{1,p}(M), f \neq 0} \frac{\int_M (|\nabla f|^p + V|f|^p)dv}{\int_M |f|^p dv},$$

and the first Neumann eigenvalue is given by

$$\lambda_{1,p}(M) = \inf_{f \in W^{1,p}(M), f \neq 0} \left\{ \frac{\int_M (|\nabla f|^p + V|f|^p)dv}{\int_M |f|^p dv}; \int_M |f|^{p-2} f dv = 0 \right\}.$$

Here $W^{1,p}(M)$ is the Sobolev space and $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in Sobolev space $W^{1,p}(M)$. The function f is then called the eigenfunction of operator L corresponding to μ (or λ) on M.

2. MAIN RESULTS

We consider a bounded domain Ω in a n-dimensional Riemannian manifold M, $n \geq 2$. Under some boundary assumption for $f : \Omega \to \mathbb{R}$, we will obtain a positive lower bound for $\mu_{1,p}$ on bounded domain Ω as follows:

Theorem 2.1. *Let Ω be a bounded domain on a Riemannian manifold M, and assume that there is a smooth function $f : \Omega \to \mathbb{R}$ such that satisfies $\|\nabla f\| \leq a$ and $\Delta_p f \geq b$ for some positive constants a, b, where $a > b$. Then the first Dirichlet eigenvalue of the quasilinear operator L satisfies*

$$\mu_{1,p} \geq \frac{b^p}{p^p a^{p(p-1)}}.$$

Proof. We first note that by density we can use smooth functions in the variational characterization of $\mu_{1,p}(\Omega)$. So given $u \in C_0^\infty(M)$, based on the fact that V is positive function, we have

$$\begin{aligned} b\int_\Omega |u|^p dv &\le \int_\Omega |u|^p(\Delta_p f + V)dv \\ &= -\int_\Omega < \nabla|u|^p, \|\nabla f\|^{p-2}\nabla f > dv + \int_\Omega |u|^p V dv \\ &= -p\int_\Omega |u|^{p-1} < \nabla|u|, \|\nabla f\|^{p-2}\nabla f > dv + \int_\Omega |u|^p V dv \\ &\le p\int_\Omega |u|^{p-1}\|\nabla u\|\|\nabla f\|^{p-1}dv + \int_\Omega |u|^p V dv \\ &\le p\int_\Omega |u|^{p-1}a^{p-1}\|\nabla u\|dv + \int_\Omega |u|^p V dv. \end{aligned} \tag{2.1}$$

Now considering a constant $c > 0$ and using Young inequality, we obtain

$$\begin{aligned} |u|^{p-1}a^{p-1}\|\nabla u\| &\le \frac{c^q|u|^{q(p-1)}}{q} + \frac{a^{p(p-1)}\|\nabla u\|^p}{pc^p} \\ &= \frac{(p-1)c^{p/(p-1)}|u|^p}{p} + \frac{a^{p(p-1)}\|\nabla u\|^p}{pc^p}. \end{aligned}$$

Therefore

$$p\int_\Omega |u|^{p-1}a^{p-1}\|\nabla u\| + \int_\Omega |u|^p V \le (p-1)c^{p/(p-1)}|u|^p + \frac{a^{p(p-1)}\|\nabla u\|^p}{c^p} + \int_\Omega |u|^p V. \tag{2.2}$$

We could choose c so that $b - (p-1)c^{p/(p-1)} = \frac{b}{p}$, that is $c^p = \frac{b^{p-1}}{p^{p-1}}$. Hence with the statement in theorem $a > b$, we know

$$\frac{p^{p-1}a^{p(p-1)}}{b^{p-1}} > 1,$$

so, (2.1) and (2.2) lead to

$$\frac{b}{p}\int_\Omega |u|^p dv \le \frac{p^{p-1}a^{p(p-1)}}{b^{p-1}}\left(\int_\Omega \|\nabla u\|^p dv + \int_\Omega |u|^p V dv\right).$$

Dividing both side to $\int_\Omega |u|^p$, completes the proof. □

As a first application, we apply this theorem for distance function:

Corollary 2.2. *Consider a bounded domain $\Omega \in M^n(c)$. If Ω is contained in a geodesic ball B_R, then*

$$\begin{aligned} \mu_{1,p}(\Omega) &\ge \frac{(n-1)^p(\sqrt{-c})^p}{p^p}\coth^p(\sqrt{-c}R), \quad if \ \ c < 0, \\ \mu_{1,p}(\Omega) &\ge \frac{(n-1)^p}{p^p R^p}, \quad if \ \ c = 0, \\ \mu_{1,p}(\Omega) &\ge \frac{(n-1)^p(\sqrt{c})^p}{p^p}\cot^p(\sqrt{c}R), \quad if \ \ c > 0. \end{aligned}$$

Note that in the above corollary $M^n(c)$ is the simply connected space form of constant sectional curvature c.

Corollary 2.3. *Let $M^n = \mathbb{R} \times N$ be a warped product Riemannian manifold endowed with the warped metric $ds^2 = dt^2 + e^{2\rho(t)}g_0$, such that the warped fuction satisfies $\rho^{'}(t) \geq \kappa > 0$, for some constant κ. Then the first Dirichlet eigenvalue of (1.1), satisfies in the following:*

$$\mu_{1,p}(M) \geq \frac{(n-1)^p}{p^p}\kappa^p.$$

Based on the studies in [2], this kind of estimate that we mentioned for warped product can be lifted for Riemannian manifolds which admite a Riemannian submersion over hyperbolic space.

Theorem 2.4. *Let $\tilde{M}^m$ be a complete Riemannian manifold that admits a Riemannian submersion $\pi : \tilde{M}^m \longrightarrow M^n = \mathbb{R} \times N$, where π is a surjective map. If the mean curvature of the fibers satisfy $\|H^{\mathcal{F}}\| \leq \alpha$, for some $\alpha < (n-1)\kappa^{1/p}$, then for the first Dirichlet eigenvalue of (1.1), we have*

$$\mu_{1,p}(M) \geq \frac{((n-1)^p\kappa - \alpha)^p}{p^p}.$$

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