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POWER BOUNDED AND MEAN ERGODIC OPERATORS ON BLOCH TYPE AND ZYGMUND TYPE SPACES

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ABSTRACT. In this paper we investigate power bounded and mean ergodic properties of composition operators on Bloch type spaces and also on Zygmund type spaces. We study their relation with a corresponding composition operator on analytic Lipschitz algebras and also on differentiable Lipschitz algebras, respectively.

1. INTRODUCTION

Let X be a Banach space and $B(X)$ denote the space of all bounded operators on X. An operator T on X is called *power bounded* if the sequence $\{T^n\}_{n\in\mathbb{N}}$ is bounded in $B(X)$. The operator T on X is called *mean ergodic* if for each $x \in X$ the sequence $\{T_{[n]}(x)\}_{n\in\mathbb{N}}$ is convergent to some $Lx \in X$ for some $L \in B(X)$, where

$$T_{[n]} := \frac{1}{n}\sum_{k=1}^{n} T^k, \qquad \text{for each } n \in \mathbb{N}.$$

Theory of mean ergodic operators on Banach spaces are related to the theory of bases in Banach spaces. It is known that a Banach space X with a basis is reflexive if and only if every power bounded operator on X is mean ergodic. This topic was first studied by Bonet and Domanski in 2011 on $H(U)$ for a Stein manifold U [1]. Later, many authors investigated power bounded

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and mean ergodic properties of different type of operators between various type of Banach spaces. In this paper we mainly focus on certain type of operators called composition operators, defined as follows.

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space all complex-valued analytic functions on $\mathbb{D}$. For an analytic selfmap φ of $\mathbb{D}$, the composition operator induced by φ, denoted by C_φ, is defined by

$$C_\varphi(f) = f \circ \varphi, \qquad \text{for each } f \in H(\mathbb{D}).$$

Composition operators appear in the study of dynamical systems and also in characterizing isometries on many analytic function spaces. Composition operators between various spaces of analytic functions have been studied by many authors, see for example [2, 3, 4] and the references therein.

Power bounded and mean ergodic composition operators on different Banach spaces have been studied by many authors, see for example [1, 5] and the references therein. In the next chapter, we investigate power bounded and mean ergodic properties of composition operators on Bloch type spaces and also on Zygmund type spaces. Indeed, we study their relation with a corresponding composition operator on analytic Lipschitz algebras and also on differentiable Lipschitz algebras, respectively.

2. Main result

For each $0 < \alpha < \infty$, the Bloch type space of order α, denoted by $\mathcal{B}^\alpha$, is the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f'(z)| < \infty.$$

The Bloch type space $\mathcal{B}^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f'(z)|, \quad (f \in \mathcal{B}^\alpha).$$

Let $A(\overline{\mathbb{D}})$ denote the classic disc algebra containing of all continuous functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$. For each $0 < \alpha \le 1$, the analytic Lipschitz algebra of order α, denoted by $Lip_A(\overline{\mathbb{D}}, \alpha)$, is algebra of all functions $f \in A(\overline{\mathbb{D}})$ with

$$\rho_\alpha(f) = \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} < \infty.$$

The analytic Lipschitz algebra $Lip_A(\overline{\mathbb{D}}, \alpha)$ is a Banach algebra with the norm

$$\|f\|_{Lip^\alpha} = \|f\|_\infty + \rho_\alpha(f), \quad (f \in Lip_A(\overline{\mathbb{D}}, \alpha)),$$

where $\|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|$.

It is known that for each $0 < \alpha < 1$, every $f \in \mathcal{B}^\alpha$ has a unique continuous extension to some $F \in Lip_A(\overline{\mathbb{D}}, 1-\alpha)$, see [3]. We next give the relation between power boundedness of a composition operator on Bloch type spaces and power boundedness of a corresponding composition operator on analytic Lipschitz algebras.

Theorem 2.1. *Let $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is power bounded if and only if the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, 1-\alpha) \to Lip_A(\overline{\mathbb{D}}, 1-\alpha)$ is power bounded, where ϕ is the unique extension of φ to $\overline{\mathbb{D}}$.*

We next give the result of Theorem 2.1 for the mean ergodicity of a composition operator between Bloch type spaces.

Theorem 2.2. *Let $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is mean ergodic if and only if the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, 1-\alpha) \to Lip_A(\overline{\mathbb{D}}, 1-\alpha)$ is mean ergodic, where ϕ is the unique extension of φ to $\overline{\mathbb{D}}$.*

Let $0 < \alpha < \infty$. The Zygmund type space of order α, denoted by $\mathcal{Z}^\alpha$, consists of those analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|)^\alpha |f''(z)| < \infty.$$

The Zygmund type space $\mathcal{Z}^\alpha$ is a Banach space, with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|)^\alpha |f''(z)|, \quad (f \in \mathcal{Z}^\alpha).$$

More generally, for each $n \in \mathbb{N}$ and $0 < \alpha < \infty$, the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|)^\alpha |f^{(n+1)}(z)| < \infty,$$

is called n-Zygmund type space and is denoted by $\mathcal{Z}^\alpha_n$, see [3]. The space $\mathcal{Z}^\alpha_n$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}^\alpha_n} := |f(0)| + |f'(0)| + \cdots + |f^{(n)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|)^\alpha |f^{(n+1)}(z)|, \quad (f \in \mathcal{Z}^\alpha_n).$$

For each $n \in \mathbb{N}$, differentiable Lipschitz algebra of order n, denoted by $Lip^n(X, \alpha)$, is the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ whose derivatives up to order n exist and $f^{(k)} \in Lip(\overline{\mathbb{D}}, \alpha)$ for each $0 \le k \le n$. The algebra $Lip^n(\overline{\mathbb{D}}, \alpha)$ is a Banach algebra equipped with the norm

$$\|f\|_{Lip^n(\overline{\mathbb{D}},\alpha)} := \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty + \rho_\alpha(f^{(k)})}{k!}, \quad (f \in Lip^n(\overline{\mathbb{D}}, \alpha)).$$

As in the case of Bloch type spaces, it is known that for each $n \in \mathbb{N}$ and $0 < \alpha < 1$, every $f \in \mathcal{Z}^\alpha_n$ has a unique continuous extension to some $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$, see [3]. Our next result gives the relation between power boundedness of a composition operator on n-Zygmund type spaces and power boundedness of a corresponding composition operator on differentiable Lipschitz algebras of order n.

Theorem 2.3. *Let $n \in \mathbb{N}$, $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is power bounded if and only* *if the composition operator* $C_\phi : Lip^n(\overline{\mathbb{D}}, 1-\alpha) \to Lip^n(\overline{\mathbb{D}}, 1-\alpha)$ *is power bounded, where* ϕ *is the unique extension of* φ *to* $\overline{\mathbb{D}}$.

As the final result, we next consider mean ergodicity of a composition operator between n-Zygmund type spaces.

Theorem 2.4. *Let* $n \in \mathbb{N}$, $0 < \alpha < 1$ *and* φ *be an analytic selfmap of* $\mathbb{D}$. *Then, the composition operator* $C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ *is mean ergodic if and only if the composition operator* $C_\phi : Lip^n(\overline{\mathbb{D}}, 1-\alpha) \to Lip^n(\overline{\mathbb{D}}, 1-\alpha)$ *is mean ergodic, where* ϕ *is the unique extension of* φ *to* $\overline{\mathbb{D}}$.

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