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ON CONVERGENCE OF ALTERNATING RESOLVENTS FOR A FINITE FAMILY OF PSEUDO-CONVEX FUNCTIONS

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Abstract. In this paper, the strong convergence of the Halpern type and weak convergence of the sequence generated by the product of resolvents of pseudo-convex functions are established in the setting of Hadamard spaces.

1. Introduction

Let (X, d) be a metric space. A geodesic from x to y is a map γ from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of a geodesic path is called a geodesic segment, which is denoted by $[x, y]$ whenever it is unique. A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic path, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic path. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$and $t \in [0, 1]$, and we write $tx \oplus (1t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = (1 - t)d(x, y) \quad and \quad d(z, y) = td(x, y).$$

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A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of

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three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\bar{\triangle} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for $i, j \in \{1, 2, 3\}$. Let $\triangle$ be a geodesic triangle in X and $\bar{\triangle}$ be a comparison triangle for $\bar{\triangle}$, then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all points $x, y \in \triangle$ and $\bar{x}, \bar{y} \in \bar{\triangle}$:

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let x, y and z be points in X and y_0 be the midpoint of the segment $[y, z]$; then, the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{1.1}$$

Inequality (1.1) is known as the CN inequality of Bruhat and Titis [2].
A geodesic space X is said to be a $CAT(0)$ space if all geodesic triangles satisfy the $CAT(0)$ inequality. Equivalently, X is called a $CAT(0)$ space if and only if it satisfies the CN inequality. $CAT(0)$ spaces are examples of uniquely geodesic spaces, and complete $CAT(0)$ spaces are called Hadamard spaces.
In a unique geodesic metric space X, a set $A \subset X$, is called convex iff for each $x, y \in A$, $[x, y] \subset A$. A function $f : X \to]-\infty, +\infty]$ is called

(1) proper iff

The domain of f defined by $D(f) := \{x \in X : f(x) < \infty\}$ is nonempty.

(2) lower semicontinuous (for short, lsc) iff

$$\{x \in D(f) : f(x) \leq r\},$$

is closed for each $r \in \mathbb{R}$.

(3) convex iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

(4) α-weakly convex for some $\alpha > 0$ iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) + \alpha\lambda(1 - \lambda)d^2(x, y).$$

(5) quasi-convex iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) \leq \max\{f(x), f(y)\}.$$

(6) pseudo-convex iff $f(y) > f(x)$ implies that there exists $\beta(x, y) > 0$ and $0 < \delta(x, y) \leq 1$ such that

$$f(y) - f((1 - \lambda)x \oplus \lambda y) \geq \lambda\beta(x, y), \quad \forall\lambda \in (0, \delta(x, y)).$$

Definition 1.1. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{x\in v} \limsup_{n\to\infty} d(v, x_n).$$

Definition 1.2. A sequence $\{x_n\}$ in a Hadamard space X is said to be weakly converges to a point $\bar{v} \in X$ if $A(\{x_n\}) = \bar{v}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $w - \lim_{n\to\infty} x_n = \bar{v}$(see [1]).

The concept of weak convergence in metric spaces was first introduced and studied by Lim [6]. Kirk and Panyanak [5] later introduced and studied this concept in $CAT(0)$ spaces and proved that it is very similar to the weak convergence in Banach space setting.

2. MAIN RESULTS

Definition 2.1. Let $\mathcal{H}$ be a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ be convex and lsc. For $\lambda > 0$, the resolvent of f of order λ at $x \in \mathcal{H}$ is defined as follows.

$$J_\lambda^f x := \mathrm{Argmin}_{y\in\mathcal{H}}\{f(y) + \frac{1}{2\lambda} d^2(x, y)\}.$$

Well-definedness of J_λ^f was proved by Jost [3] and Mayer [7]. In [4, Theorem 3.1] the authors proved that for an α-weakly convex function f, the resolvent $J_\lambda^f x$ exists for all $x \in \mathcal{H}$ and $\lambda < \frac{1}{2\alpha}$.

Theorem 2.2. *Suppose that $\mathcal{H}$ is a locally compact Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are proper, pseudo-convex functions and lsc. If $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$. Then the sequence generated by*

$$x_{k+1} = J_{\lambda_k}^{f_N} \cdots J_{\lambda_k}^{f_1} x_k \tag{2.1}$$

converges to an element of $\cap_{i=1}^N \mathrm{Argmin}(f_i)$

Theorem 2.3. *Suppose that $\mathcal{H}$ is a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are proper, convex and lsc. If $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$ then the sequence (2.1) converges weakly to an element of $\cap_{i=1}^N \mathrm{Argmin}(f_i)$*

Halpern regularization of (2.1) gets a strong convergence theorem.

Theorem 2.4. *Suppose that $\mathcal{H}$ is a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are α-weakly convex, quasi-convex and lsc. Let $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$, $u \in \mathcal{H}$ is arbitrary and α_k satisfies the conditions*

- $0 \leq \alpha_k \leq 1$
- $\alpha_k \to 0 \quad as \quad k \to \infty$
- $\sum_{k=1}^\infty \alpha_k = +\infty$

If f_i for each $(1 \leq i \leq N)$ are pseudo-convex functions then the sequence generated by

$$x_{k+1} = \alpha_k u \oplus (1 - \alpha_k) J_{\lambda_k}^{f_N} \cdots J_{\lambda_k}^{f_1} x_k$$

converges strongly to an element $\cap_{i=1}^{N} \mathrm{Argmin}(f_i)$

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