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FIXED POINT THEOREMS FOR CYCLIC WEAK CONTRACTIONS IN MODULAR METRIC SPACES

HOSSEIN RAHIMPOOR*

Department of Mathematics, Payame Noor University P. O. Box 19395-3697, Tehran, Iran
rahimpoor2000@pnu.ac.ir

Abstract. The purpose of this paper is to present some fixed point results for ϕ-contractions in modular metric spaces.

1. Introduction

The concept of modular spaces was introduced by Nakano [10] and was later reconsidered in detail by Musielak and Orlicz [8, 9]. In 2010, Chistyakov [2] introduced a new metric structure, which has a physical interpretation and generalized modular spaces and complete metric spaces by introducing modular metric spaces. For more features of concepts of modular metric spaces, see e. g., [1, 3, 4]. Fixed point theory involves many fields of mathematics and branches of applied science such as functional analysis, mathematical analysis, general topology and operator theory. In 2003, Kirk et al. [7] introduced cyclic contraction in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mapping. Later, Karapinar and Erhan[6] proved the existence of fixed points for various types of cyclic contractions in a metric space. Recently, E. Karapinar in [5] proves a fixed point theorem for an operator T on a complete metric space X when X has a cyclic representation with respect to T. In this paper, we improve and generalized the fixed point results for mappings

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* Speaker.

satisfying cyclical contractive conditions established by E. Karapinar [5], in modular metric spaces.

Definition 1.1. Let X be an arbitrary set, the function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ that will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $x,y \in X$ and for all $\lambda > 0$, is said to be a modular metric on X (or simply a modular if no ambiguity arises) if it satisfies the following three conditions:
(i) given $x,y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and nonly if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$ and $x,y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda,\mu > 0$ and $x,y,z \in X$.
If instead of (i), we have only the condition:
(i_1) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i_1) and
(i_2) given $x,y \in X$, if there exists $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = 0$ implies that $x = y$, then ω is called a *strict modular* on X.

If instead of (iii) we replace the following condition for all $\lambda,\mu > 0$ and $x,y,z \in X$;

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\lambda(z,y), \tag{1.1}$$

then ω is called a *convex modular* on X.

Definition 1.2. [2] Given a modular ω on X, the sets

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x,x_\circ) \to 0 \ as \ \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_\circ) = \{x \in X : \omega_\lambda(x,x_\circ) < \infty \ \ for\ some \ \ \lambda > 0\}$$

are said to be modular spaces (around $x_\circ$). Also the modular spaces X_ω and X^*_ω can be equipped with metrics d_ω and d^*_ω, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\}, \quad x,y \in X_\omega$$

and

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\}, \quad x,y \in X^*_\omega$$

If ω is a convex modular on X, then according to [2, Theorem 3.6] the two modular spaces coincide, $X_\omega = X^*_\omega$.

Definition 1.3. Given a modular metric space X_ω, a sequence of elements $\{x_n\}_{n=1}^\infty$ from X_ω is said to be modular convergent ($\omega-$convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x , such that $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$. This will be written briefly as $x_n \overset{\omega}{\to} x$, as $n \to \infty$.

Definition 1.4. [4] A sequence $\{x_n\} \subset X_\omega$ is said to be ω-*Cauchy* if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m,n\to\infty} \omega_\lambda(x_n,x_m) = 0$, i.e.,

$$\forall\varepsilon > 0 \ \ \exists \ \ n_\circ(\varepsilon) \in \mathbb{N} \ \ such \ \ that \ \ \forall n,m \geq n_\circ(\varepsilon) \ \ : \ \ \omega_\lambda(x_n,x_m) \leq \varepsilon.$$

Modular metric space X_ω is said to be *ω-complete* if each ω-Cauchy sequence from X_ω be modular convergent to an $x \in X_\omega$.

Remark 1.5. A modular $\omega = \omega_\lambda$ on a set X, for given $x, y \in X$, is non-increasing on λ. Indeed if $0 < \lambda < \mu$, then we have

$$\omega_\mu(x,y) \leq \omega_{\mu-\lambda}(x,x) + \omega_\lambda(x,y) = \omega_\lambda(x,y)$$

for all $x, y \in X$.

2. Main result

Definition 2.1. Let X_ω be a modular metric space, $p \in \mathbb{N}$, and $T : X_\omega \to X_\omega$ a map. Then we say that $\cup_{i=1}^{p} A_i$ (where $A_i \subseteq X_\omega$ for all $i \in \{1, 2, ..., p\}$) is a cyclic representation of X with respect to T if and only if the following two conditions hold:
(I) $X_\omega = \cup_{i=1}^{p} A_i$;
(II) $T(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq p-1$, and $T(A_p) \subseteq A_1$.

Definition 2.2. Let X_ω be a modular metric space, m a positive integer, $A_1, A_2, ..., A_m$ ω-closed nonempty subset of $X_\omega$and $Y = \cup_{i=1}^{m} A_i$ and $T : Y \to Y$ an operator. T is called a cyclic weak ϕ-contraction if
(I) $\cup_{i=1}^{m} A_i$is a cyclic representation of Y with respect to T;
(II) there exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$, such that

$$\omega_\lambda(Tx, Ty) \leq \omega_\lambda(x,y) - \phi(\omega_\lambda(x,y)) \tag{2.1}$$

for all $\lambda > 0$ and for any $x \in A_i, y \in A_{i+1}$, $i = 1, 2, ..., m$ where $A_{m+1} = A_1$.

Example 2.3. Let $X_\omega = [0,1]$, we take a mapping $\omega : (0, \infty) \times [0,1] \times [0,1] \to [0, \infty]$ which is defined by $\omega_\lambda(x,y) = \dfrac{|x-y|}{\lambda}$ for all $x, y \in X = X_\omega$ and $\lambda > 0$. Consider the ω-closed nonempty subsets of X_ω as follow : $A_1 = [0,1], A_2 = [0, \frac{2}{3}], A_3 = [0, \frac{1}{2}], A_4 = [0, \frac{5}{12}], A_5 = [0, \frac{3}{8}]$ with $X_\omega = Y = \cup_{i=1}^{5} A_i$. Let $T : X_\omega \to X_\omega$ be the mapping defined by $Tx = \dfrac{3x+1}{6}$. Then, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_4, T(A_4) \subseteq A_5, T(A_5) \subseteq A_1$. And

$$\omega_\lambda(Tx, Ty) = \frac{|\frac{3x+1}{6} - \frac{3y+1}{6}|}{\lambda} = \frac{1}{\lambda}(\frac{|x-y|}{2}) \leq \omega_\lambda(x,y) - \frac{1}{2}\omega_\lambda(x,y).$$

Furthermore, if $\phi : [0, \infty) \to [0, \infty)$ is defined by $\varphi(t) = \frac{t}{2}$, then ϕ is strictly increasing and T is a cyclic weak ϕ-contraction.

Remark 2.4. Rewriting the inequality 1.1 in the form

$$(\lambda + \mu)\omega_{\lambda+\mu}(x,y) \leq \lambda\omega_\lambda(x,z) + \mu\omega_\mu(y,z)$$

we find that the function ω is a convex modular on X if and only if the function $\widehat{\omega}(x,y) = \lambda\omega_\lambda(x,y)$ for all $\lambda > 0$ and $x, y \in X$, is simply a modular on X, and the function $\lambda \mapsto \widehat{\omega}(x,y) = \lambda\omega_\lambda(x,y)$ are non-increasing on $(0,\infty)$:

$$\text{if } 0<\lambda\le\mu, \text{ then } \omega_\mu(x,y)\le\frac{\lambda}{\mu}\omega_\lambda(x,y)\le\omega_\mu(x,y).$$

Now, for any $\mu\ge\lambda$ we find $k\in\mathbb{R}_+$ such that $\mu=k\lambda$ and so

$$\omega_{k\lambda}(x,y)\le\frac{1}{k}\omega_\lambda(x,y). \tag{2.2}$$

Theorem 2.5. *Let ω be a convex modular on X such that X_ω is a $\omega-$complete modular metric space, m is a positive integer, $A_1, A_2, ..., A_m$ ω-closed subsets of X_ω and $Y=\cup_{i=1}^m A_i$. Suppose that $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)$ is a non-decreasing function and $\varphi(t)=0$ only for $t=0$ and $T: X_\omega\times X_\omega\to X_\omega$ is a cyclic weak φ-contraction where $Y=\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T. Then, T has a unique fixed point $z\in\cap_{i=1}^m A_i$.*

Theorem 2.6. *Let $T:Y\to Y$ be a self mapping as in Theorem 2.5 .*
(i) If there exists a sequence $\{y_n\}$ in Y with $\lim_{n\to\infty}\omega_\lambda(y_n,Ty_n)=0$ then $\lim_{n\to\infty}\omega_\lambda(y_n,z)=0$.
(ii) If there exists a ω-convergent sequence $\{y_n\}$ in Y with $\lim_{n\to\infty}\omega_\lambda(y_{n+1},Ty_n)=0$ then there exists $x\in Y$ such that $\lim_{n\to\infty}\omega_\lambda(y_n,T^n x)=0$.

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