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APPROXIMATELY JENSEN-HOSSZU $\rho-$FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce the concept of the Jensen-Hosszu $\rho-$functional equations between Banach algebras and we investigate it as an additive equation. Also, we prove the Hyers-Ulam stability of Jensen-Hosszu $\rho-$functional equations between Banach algebras.

1. Introduction

In 1940, Ulam[10] presented some unsolved problems, and among them posed the following question. "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? " Ulam raised the stability of functional equations and Hyers[2] in 1941 was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces.
In 1967, M. Hosszu introduced the functional equation $f(x + y - xy) = f(x) + f(y) - f(xy)$ in a presentation at a meeting on functional equations held in Zakopane, Poland. In honor of M. Hosszu, this equation is called Hosszu's functional equation. As one can easily see, Hosszu's functional equation is a kind of generalized form of the $f(x + y) = f(x) + f(y)$ functional equation. In 1996, L. Losonczi [9] proved the Hyers-Ulam stability of

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the Hosszu equation in the class of real functions and expressed the following theorem.

Theorem 1.1. *(L. Losonczi) Let E be a Banach space and suppose that $f : \mathbb{R} \to E$ satisfies the inequality*

$$\|f(x+y-xy) - f(x) - f(y) + f(xy)\| \leq \delta \quad (x, y \in \mathbb{R})$$

for some $\delta \geq 0$. Then there exists an additive function $T : \mathbb{R} \to E$ and a unique constant $b \in E$ such that

$$\|f(x) - T(x) - b\| \leq 20\delta$$

for all $x \in \mathbb{R}$.

Many authors have searched the stability of Cauchy, Jensen and Hosszu equation based on the concept of Hyers-Ulam stability (see [1, 3, 4, 5, 6, 7, 8]).

In the following we defined Jensen-Hosszu $\rho-$functional equation on Banach algebras.
Let $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach algebras, let a mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies

$$f(x+y-xy) + f(xy) - 2f(\frac{x+y}{2}) = \rho\Big(f(x+y-xy) + f(xy) - f(x+y)\Big) \tag{1.1}$$

and

$$f(x+y-xy) + f(xy) - f(x+y) - \rho\Big(f(x+y-xy) + f(xy) - 2f(\frac{x+y}{2})\Big) \tag{1.2}$$

where $\rho \neq 0, \pm 1$ is a fixed real number and for all $x, y \in \mathfrak{A}$, then we called Jensen-Hosszu $\rho-$functional equation.
In this work, we solve the functional equations of the form (1.1) and (1.2) in the class of real functions as an additive equation and prove them with the above ideas theorems have stable in the Hyers–Ulam sense.

2. Stability of Jensen-Hosszu $\rho-$functional equation

In this section, let $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach algebras. Firstly, in the next lemma, we solve that f is an additive mapping.

Lemma 2.1. *If a mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies*

$$f(x+y-xy)+f(xy)-2f(\frac{x+y}{2}) = \rho\Big(f(x+y-xy)+f(xy)-f(x+y)\Big) \tag{2.1}$$

for all $x, y \in \mathfrak{A}$ and $\rho \neq 0, \pm 1$ is a fixed real number, then the mapping f is an additive equation.

In the following theorem, the functional equation (1.1) can be stable in Banach algebras.

Theorem 2.2. *Let $f:\mathfrak{A}\to\mathfrak{A}$ be a mapping such that*

$$\Big\|f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})-\rho\Big(f(x+y-xy)+f(xy)-f(x+y)\Big)\Big\|\le\delta \tag{2.2}$$

where $\rho\neq 0,\pm 1$ is a fixed real number, for some $\delta\geq 0$ and for all $x,y\in\mathfrak{A}$. Then there exists a unique additive $T:\mathfrak{A}\to\mathfrak{A}$ such that

$$\|f(x)-T(x)\|\le\frac{1}{2}\delta$$

for all $x\in\mathfrak{A}$.

In the next lemma, we solve that f is an additive mapping.

Lemma 2.3. *If a mapping $f:\mathfrak{A}\to\mathfrak{B}$ satisfies*

$$f(x+y-xy)+f(xy)-f(x+y)=\rho\Big(f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})\Big) \tag{2.3}$$

for all $x,y\in\mathfrak{A}$ and $\rho\neq 0,\pm 1$ is a fixed real number, then the mapping f is an additive equation.

In the following theorem, we investigate Hyers-Ulam staility of functional equation (1.2) in Banach algebras.

Theorem 2.4. *Let $f:\mathfrak{A}\to\mathfrak{A}$ be a mapping such that*

$$\Big\|f(x+y-xy)+f(xy)-f(x+y)-\rho\Big(f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})\Big)\Big\|\le\delta \tag{2.4}$$

where $\rho\neq 0,\pm 1$ is a fixed real number, for some $\delta\geq 0$ and for all $x,y\in\mathfrak{A}$. Then there exists a unique additive $T:\mathfrak{A}\to\mathfrak{A}$ such that

$$\|f(x)-T(x)\|\le\delta$$

for all $x\in\mathfrak{A}$.

3. Conclusions

We introduced the new concept of Jensen-Hosszu ρ-functional equations in Banach algebras and in the lemmas we investigated it as an equal to an additive equation and in the main theorems, we proved that the Jensen-Hosszu ρ-functional equations can be stable in Banach algebras.

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