The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

ORTHOGONALLY GENERALIZED JENSEN-TYPE $\rho-$FUNCTIONAL EQUATION

MOHAMMAD TAGHI HEYDARI* AND VAHID KESHAVARZ

Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75918 Iran
heydari@yu.ac.ir and v.keshavarz68@yahoo.com

Abstract. In this paper, we introduce and solve the concept of the generalized Jensen-type $\rho-$functional equation. Finally, we investigate the Hyers-Ulam stability of generalized Jensen-type $\rho-$functional equation with Găvruta's control function on orthogonally Banach algebras approach direct methods.

1. Introduction

A classical question in the sense of functional equation says that "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? " Ulam raised the stability of functional equations and Hyers was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces. Th. M. Rassias proved a generalized version of the Hyers's theorem for approximately additive maps. Găvruta generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference general control function $\varphi(x, y)$. The study of stability problem of functional equations have been done by several authors on different spaces such as Banach, C^*-Banach algebras and modular spaces (for example see [2, 3, 4, 6]).

2020 *Mathematics Subject Classification. Primary 39B52; Secondary 39B82, 22D25*

Key words and phrases. Hyers-Ulam stability, orthogonally generalized Jensen-type $\rho-$functional equation, orthogonally Banach algebras.

* Sspeaker.

Recently, Eshaghi Gordji et al. [1] introduced notion of the orthogonal. The study on orthogonal sets has been done by several authors (for example, see [5, 7, 8])

Definition 1.1. [1] Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If there exists $u_0 \in X$ such that for all $v \in X$,

$$v \perp u_0 \ \ or \ \ u_0 \perp v,$$

then $\perp$ is called an orthogonally set (briefly O-set). We denote this O-set by $(X, \perp)$. Let $(X, \perp)$ be an O-set and (X, d) be a generalized metric space, then $(X, \perp, d)$ is called orthogonally generalized metric space.

Let $(X, \perp, d)$ be an orthogonally metric space.
(i) A sequence $\{u_n\}_{n\in\mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if for any $n \in \mathbb{N}$,

$$u_n \perp u_{n+1} \ \ or \ \ u_{n+1} \perp u_n.$$

(ii) Mapping $f : X \to X$ is called $\perp-$continuous in $u \in X$ if for each O-sequence $\{u_n\}_{n\in\mathbb{N}}$ in X with $u_n \to u$, then $f(u_n) \to f(u)$. Clearly, every continuous map is $\perp-$continuous at any $u \in X$.
(iii) $(X, \perp, d)$ is called orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent to a point in X.
(iv) Mapping $f : X \to X$ is called $\perp$-preserving if for all $u, v \in X$ with $u \perp v$, then $f(u) \perp f(v)$.
Consider the orthogonally generalized Jensen-type ρ-functional equation

$$\begin{aligned}
&f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)= \\
&\rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big) \\
&+f(u)+f(v)+f(w)\Big)
\end{aligned} \tag{1.1}$$

such that $\rho \neq 0, \pm 1$ is a real number and $u \perp v,\ u \perp w,\ v \perp w$.
In this paper, we investigate (1.1) is additive equation and the Hyers-Ulam of it's equation approach direct methods with Găvruta's control function.

2. Hyers-Ulam Stability

Throughout this section, let A and B are two orthogonally Banach algebras.
To prove the main theorem, we need the following lemma. Firstly, in the next lemma, we prove that f is an additive mapping.

Lemma 2.1. *If a mapping $f : A \to B$ satisfies*

$$f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)=$$
$$\rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big)$$
$$+f(u)+f(v)+f(w)\Big) \tag{2.1}$$

for all $u, v, w \in A$ with $u \perp v$, $u \perp w$, $v \perp w$, then the mapping f is additive.

In the following theorem, we prove Hyers-Ulam stability of orthogonally generalized Jensen-type ρ-functional with Găvruta's control function on orthogonally Banach algebras.

Theorem 2.2. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(u,v,w):=\sum_{n=0}^{\infty}\frac{1}{2^n}\varphi(2^n u,2^n v,2^n w)<\infty \tag{2.2}$$

Suppose that $f : A \to B$ is a mapping satisfying

$$\Big\|f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)-$$
$$\rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big)$$
$$+f(u)+f(v)+f(w)\Big)\Big\|\le\varphi(u,v,w) \tag{2.3}$$

for all $u, v, w \in A$ with $u \perp v$, $u \perp w$, $v \perp w$. Then there exist a unique additive mapping $T : A \to B$ such that

$$\|f(u)-T(u)\|\le\frac{1}{3}\varphi(u,v,w)$$

for all $u \in A$.

In the next corollary we prove the Hyers-Ulam stability of orthogonally generalized Jensen-type ρ-functional with Rassias's control function on orthogonally Banach algebras.

Corollary 2.3. *Let θ, p_i, q_i, $i = 1, 2, 3$ are positive real such that $p_i < 1$ and $q_i < 3$. Suppose that $f : A \to B$ is a mapping such that*

$$\Big\|f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)-$$
$$\rho\Big(3\big(f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big)$$
$$+f(u)+f(v)+f(w)\Big)\Big\|\le\theta(\|u\|^{p_1}+\|v\|^{p_2}+\|w\|^{p_3}), \tag{2.4}$$

for all $u, v, w \in A$ *with* $u \perp v,\ u \perp w,\ v \perp w$. *Then there exist a unique additive mapping* $T : A \to B$ *such that*

$$\|f(u) - T(u)\| \leq \frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|u\|^{p_1} + \frac{1}{2-2^{p_2}}\|u\|^{p_2} + \frac{1}{2-2^{p_3}}\|u\|^{p_3}\}$$

for all $u \in A$.

References

1. M. Eshaghi Gordji, M. Ramezani, M. De La Sen and Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, **18** (2017) 569–578.
2. S. Jahedi and V. Keshavarz, *Approximate generalized additive-quadratic functional equations on ternary Banach algebras,* Journal of Mathematical Extension, **16** (2022) 1-11.
3. V. Keshavarz and S. Jahedi, *Fixed Points and Lie Bracket (Ternary) Derivation-Derivation,* The Journal of Analysis, (2022) 1-11.
4. V. Keshavarz and S. Jahedi, *Hyperstability of Orthogonally 3-Lie Homomorphism: An Orthogonally Fixed Point Approach,* In: Daras, N.J., Rassias, T.M. (eds) Approximation and Computation in Science and Engineering. Springer Optimization and Its Applications, **180** (2022). Springer, Cham.
5. V. Keshavarz and S. Jahedi, *Orthogonally* C^**-ternary Jordan homomorphisms and Jordan derivations: Solution and stability,* Journal of Mathematics, 2022, Article ID 3482254, 7 pages. https://doi.org/10.1155/2022/3482254
6. V. Keshavarz, S. Jahedi and M. Eshaghi Gordji, *Ulam-Hyers Stability of* C^**-Ternary 3-Jordan Derivations,* South East Aisian Bull. Math., **45** (2021) 55-64.
7. V. Keshavarz, S. Jahedi, M. Eshaghi Gordji and S. Bazeghi, *Hyperstability of Orthogonally Pexider Lie Functional Equation: An Orthogonally Fixed Point Approach,* Thai Journal of Mathematics, **20** (2022) 235-243.
8. M. Ramezani and H. Baghani, *Some new stability results of a Cauchy-Jensen equation in incomplete normed spaces,* Journal of Mathematical Analysis and Applications, **495** (2021).