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SOME MINIMAX THEOREMS FOR VECTOR-VALUED FUNCTIONS IN G-CONVEX SPACES

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Abstract. In this article, we give some generalized minimax inequalities for vector-valued functions by means of the generalized KKM theorem.
Keywords: G-convex space, Generalized KKM Map, Cone-γ-generalized quasi-convex (concave), Minimax Inequality.

1. INTRODUCTION

The minimax inequality plays a significant role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory and fixed point theory. Because of widespread use, this inequality has been extended in variety of ways. (For example, see M. Salehnejad and M. Azhini [8], Ding and Tan [3], Horvath [7], Georgiev and Tanaka [6])
At the beginning, the consideration of minimax theorems were mainly devoted to the study of real and vector-valued functions in topological vector spaces. Motivated by the well-known works of Horvath [7], there here appeared many generalizations of the concept of convex subset of a tapological vector space. The most general one seems to be that of the generalized

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convex space or G-convex space introduced by Park and Kim [10, 11] which extends many generalizd convex structures on topological vector space. This will be the framework in which we obtain in this work some minimax inequalities for vector-valued functions. (See Chen [2], Chang et al [1], M.G.Yang et al [14])

2. Preliminaries

Let X be a topological space and E be a nonempty subset of X. We denote by $\langle E\rangle$, the family of all nonempty finite subsets of E. Let Δ_n be the standard n–simplex $(e_1, \dots, e_n)$ in R^{n+1}. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j, j \in J\}$. The following notion of a generalized convex (or G–convex) space was introduced by Park and Kim [12]. Let X be a topological space and D is a nonempty set, $(X, D; \Gamma)$ is said to be a G–convex space if for each $A = \{a_0, \dots, a_n\} \in \langle D\rangle$, there exists a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset A$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.
When $D \subset X$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$ and if $X = D$, we write $(X; \Gamma)$ in place of $(E, E; \Gamma)$. For a G–convex space $(X \supset D; \Gamma)$,

(1) a subset Y of X is said to be Γ–convex if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$;
(2) the Γ–convex hull of a subset Y of X, denoted by $\Gamma\ -(Y)$, is defined by

$$\Gamma\ -(Y) = \bigcap\{Z \subset X : Z \text{ is a } \Gamma\text{–convex subset containing } Y\ \}.$$

Definition 2.1. If V is a real vector space, a nonempty subset $P \subset V$ is a cone if for every $x \in P$ and for every $\lambda \geq 0$, we have $\lambda x \in P$. The cone P is called

(1) convex if for all $x_1, x_2 \in P$, $x_1 + x_2 \in P$
(2) pointed if $P \cap (-P) = \{0\}$
(3) proper if $P \neq \{0\}$ and $P \neq V$
(4) solid if $\text{int}P \neq \emptyset$ (where $\text{int}P$ denotes the interior of the set P).

If P is a convex cone of a real vector space V, the relation $\preceq_p$ define below is a (partial) vector ordering of V:

$$x \preceq_p y \ \Leftrightarrow\ y - x \in P,\ \forall x, y \in V$$

Definition 2.2. Let Y be a nonempty set and E be a nonempty subset of a G–convex space $(X, D; \Gamma)$. $T : Y \to 2^E$ is called a generalized KKM mapping if for any finite set $\{y_0, y_1, \dots, y_n\} \subset Y$, there exists $\{x_0, x_1, \dots, x_n\} \in \langle E \cap D\rangle$ such that for any subset $\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\} \subset \{x_0, \dots, x_n\}$, $0 \leq k \leq n$, we have

$$\Gamma\left(\{x_{i_0}, \dots, x_{i_k}\}\right) \subset \bigcup_{j=0}^{k} T(y_{i_j})$$

Theorem 2.3 ([8])**.** *Let E be a nonempty Γ–convex subset of a G–convex space $(X;\Gamma)$ and $G : E \to 2^X$ be such that for any $y \in E$, $G(y)$ is compactly closed. Then:*

(1) *If G is a generalized KKM mapping, then the family of sets $\{G(y) : y \in E\}$ has the finite intersection property.*
(2) *If the family $\{G(y) : y \in E\}$ has the finite intersection property and $\Gamma(x) = \{x\}$ for each $x \in X$, then G is a generalized KKM mapping.*

Definition 2.4. Let Y be a nonempty set and $(X;\Gamma)$ be a G-convex space and E, F are nonempty subsets of X, Y, respectively. The bi-function $\varphi : E \times F \to V$ is said to be cone-γ-generalized quasi-convex (concave) in second component for some $\gamma \in V$, if for any finite subset $\{y_0, y_1, \ldots, y_n\} \subset F$, there exists a finite subset $\{x_0, x_1, \ldots, x_n\} \subset E$ such that for any subset $\{x_{i_0}, ..., x_{i_k}\} \subset \{x_0, x_1, \ldots, x_n\}$ and any $x^* \in co_{\mathcal{C}}\{x_{i_0}, ..., x_{i_k}\}$, there exists $j \in \{0, ..., k\}$ such that

$$\varphi(x^*, y_{i_j}) \in \gamma + P$$
$$(\ \varphi(x^*, y_{i_j}) \in \gamma - P\ ).$$

3. Main Results

In this section, we present Minimax inequalities in G-convex spaces for vector-valued functions. In sequel, suppose that X is a Hausdorff topological space, $(X;\Gamma)$ is a G–convex space and E, F are nonempty Γ–convex subsets of X. Also, (V, P) is an ordered topological vector space and P is a closed pointed convex cone such that $P \neq \phi$.

Definition 3.1. A function $\varphi : X \longrightarrow V$ is called lower [resp. upper] semi-continuous if for every $\gamma \in V$, the set $\{x \in X : \ f(x) \in \gamma - P\}$ [resp. $\{x \in X : \ f(x) \in \gamma + P\}$] is closed in X. [9]

Theorem 3.2. *Let φ and ψ be two functions from $E \times F$ to V such that:*

(1) *$\varphi(x, y)$ is lower semi-continuous in x, for each $y \in F$;*
(2) *$\psi(x, y)$ is cone–γ–generalized quasi-concave in y, for some $\gamma \in V$;*
(3) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times F$;*
(4) *for some $y_0 \in F$, $\{x \in E \ : \ \varphi(x, y_0) \in \gamma - P\}$ is a compact subset of E.*

Then there exists an $\bar{x} \in E$ such that

$$\varphi(\bar{x}, y) \in \gamma - P \text{ for all } y \in F.$$

Remark 3.3. Fan's Minimax inequality can be deduced from the above Theorem if $X = Y$, $E = F$, $V = \mathbb{R}$ and $\varphi = \psi$. (See [4])

Theorem 3.4. *Let $\varphi : E \times E \to V$ and $\gamma \in V$ be such that*

(1) *for each $x \in E$, $\varphi(x, y)$ is a lower semi-continuous function of y on each non-empty compact subset C of E;*
(2) *$\varphi(x, y)$ is cone–γ–generalized quasi-concave in x;*

(3) *there exist a non-empty compact convex subset M of E and a non-empty compact K of E such that for each $y \in E \setminus K$, there is an $x \in M$ with $\varphi(x, y) \notin \gamma - P$.*

Then there exists $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \in \gamma - P$ for all $x \in E$.

As an immediate result of Theorem 3.4, we can conclude the following minimax inequality theorem, which in turn generalizes minimax inequality due to Ding and Tan [3] and Fan [5], to vector-valued and cone$-\gamma-$generalized quasi-concave functions.

Theorem 3.5. *Let $\gamma \in V$ and $\varphi, \psi : E \times E \to V$ be two functions such that*

(1) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times E$*
(2) *for each fixed $x \in E$, $\varphi(x, y)$ is a lower semi-continuous function of y on each non-empty compact subset C of E;*
(3) *$\psi(x, y)$ is cone$-\gamma-$generalized quasi-concave in x;*
(4) *there exist a non-empty compact convex subset M of E and a non-empty compact subset K of E such that for each $y \in E \setminus K$, there is an $x \in M$ with $\varphi(x, y) \notin \gamma - P$.*

Then there exists $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \in \gamma - P$ for all $x \in E$.

The following is a generalization of minimix inequality due to Tan and Yuan [13], to vector-valued and cone$-\gamma-$generalized quasi-concave functions.

Theorem 3.6. *Let $\gamma \in V$ and $\varphi, \psi : E \times F \to V$ be two functions such that*

(1) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times F$*
(2) *for each fixed $y \in F$, $\varphi(x, y)$ is a lower semi-continuous function of x on each non-empty compact subset C of E;*
(3) *$\psi(x, y)$ is cone$-\gamma-$generalized quasi-concave in y;*
(4) *there exists a non-empty compact subset K of E and $y^* \in F$ such that $\psi(x, y^*) \notin \gamma - P$ for each $x \in E \setminus K$.*

Then there exists an $\hat{x} \in K$ such that $\varphi(\hat{x}, y) \in \gamma - P$ for all $y \in F$.

References

1. S. S. Chang, G. M. Lee and B. S. Lee, *Minimax inequalities for vector-valued*, mappings on W-spaces, J. Math. Anal. Appl., **198** (1996) 371-380.
2. G. Y. Chen, *A generalized section theorem and a minimax inequality for a vector-valued mapping*, Optimization, **22** (1991), 745-754.
3. X.P. Ding, K.K. Tan, *A minimax inequality with applications to existence of equilibrium point*, and fixed point theorems *Colloq*. Math., **63** (1992), 233-247.
4. K. Fan, *A minimax inequality and application*, *in Inequalities (O. Shisha, Ed.)*, Vol. **3**, pp. 103 113, Academic Press, New York, 1972.
5. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519 537.
6. P.G. Georgiev, T. Tanaka, *Fan's inequality for set-valued maps*, Nonlinear Anal, **47** (2001), 607-618.

7. C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341-357.
8. M. Salehnejad and M. Azhini, *Vector Variational Inequalities In G-Convex Space*, Tamkang Journal of Mathematics, Volume **53**, 2022.
9. N. S. Parageorgiou, *Nonsmooth analysis on partially ordered vector spaces.* 1. Convex case, Pacific J. Math. **107** (1983), 403-458.
10. S. Park, *Elements of the KKM theory on abstract convex spaces*, Journal of the Korean Mathematical Society, vol. **45**, no.1, pp. 1 27, 2008.
11. S. Park, *On the Von Neumann-Sion minimax theorem in KKM spaces*, Appl. Math. Lett. **23** (2010), 1269 1273.
12. S. Park, and H. Kim, *Foundations of the KKM theory on generalized convex spaces* J. Math. Anal. App, **209** (1997), 551-571.
13. K. K. Tan and X. Z. Yuan, *A minimax inequality with applications to existence of equilibrium points*, Bull. Austral. Math. Soc. **47** (1993), 483 503.
14. M. G. Yang, J. P. Xu, N. J. Huang, and S. J. Yu, *Minimax theorems for vector-valued mappings in abstract convex spaces*, Taiwanese Journal of Mathematics, vol. **14**, no. 2, pp. 719 732, 2010.