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ON DIFFERENTIAL SUBORDINATION FOR ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENT

MOSTAFA AMANI

Faculty of Science, Department of Mathematics, Urmia University, Urmia, Iran
mos.amani@urmia.ac.ir , mos_amani@yahoo.com

Abstract. Some new results closely related to the generalized Briot-Bouquet differential subordination are investigatedin in a new approach for functions with fixed second coefficient.

1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let define two well-known classes of Analytic functions as follows.

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \cdots , z \in \mathbb{U}\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1}z^{n+1}, z \in \mathbb{U}\}.$$

We denote by $\mathcal{A} = \mathcal{A}_1$ and let $\mathcal{S} \subset \mathcal{A}$ be the class of Univalent functions. As we know, class $\mathcal{S}^*$, set of Starlike functions and class $\mathcal{C}$, set of Convex functions are defined by

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \mathfrak{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \ , \ z \in \mathbb{U} \right\},$$

$$\mathcal{C} = \left\{ f \in \mathcal{A} : \mathfrak{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \ , \ z \in \mathbb{U} \right\}.$$

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Also, we denote by $\mathcal{H}_\beta[a,b]$ and $\mathcal{A}_{n,b}$ sets of analytic functions with fixed initial coefficient, respectively, as below:

$$\mathcal{H}_\beta[a,n] = \left\{p \in \mathcal{H} : p(z) = a + \beta z^n + a_{n+1}z^{n+1} + \cdots \right\},$$

and

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + \cdots , z \in \mathbb{U}\}.$$

where β and $b \in \mathbb{C}$ are fixed. Here, we assume that β and b are positive real numbers.

The concept of subordination was introduced to describe a relation between pairs of analytic functions; Let $f(z)$ and $g(z)$ be members of the class $\mathcal{H}$. we sayvthat $f(z)$ is subordinate to $g(z)$ and write by $f(z) \prec g(z)$ if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0, |w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. It is easy to see that when $g(z)$ is univalent in $\mathbb{U}$, then

$$f(0) = g(0) \quad and \quad f(\mathbb{U}) \subseteq g(\mathbb{U}),$$

is the equivalent definition of subordination.

The beginning of differential subordination theory began in 1974 by Miller, Mocanu and Reade [7]. First they proved the following result: if α is real and p is analytic in the unit disk $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}$$

implies $p(z) \prec (1 + z)/(1 - z)$, which implies properties of the range of a functions from the range of a combination of the derivative of the functions. Then in 1981, Miller and Mocanu [6] introduced the analogues differential subordination and built the theory for this type of differential implications.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, h be univalent in $\mathbb{U}$. A function p is analytic in $\mathbb{U}$ and called a solution of the differential subordination if it satisfies the (second order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h. \tag{1.1}$$

The univalent function q is called a dominant of the solutions of the differential subordination, or simply a dominant, if $p \prec q$ for all p satisfying (1.1).

In 2011, Rosihan, Nagpal and Ravichandran [12] extended the theory of second-order differential subordination for functions with fixed initial coefficient. This led to many results related to the differential subordination being extended and improved, that recently have published several articles on the application of this new result(For example, see [1, 2, 3, 4]).

In this paper, by extension of the Nunokawa lemma [9, 10] due to author et al. [2], some new results closely related to the generalized Briot-Bouquet differential subordination are investigated in a new approach for functions with fixed second coefficient. First, we need some of the following fundamental definition and theorems.

Definition 1.1. ([8], [p.24]) Assume that $\mathbf{Q}$ is the set of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$ with

$$E(q) := \left\{ \zeta \in \ \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty \right\}.$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$.

Lemma 1.2. [12] *Let $q \in \mathbf{Q}$ with $q(0) = a$, and $p \in \mathcal{H}_\beta[a, n]$ with $p(z) \not\equiv a$. If $p \not\prec q$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\setminus E(q)$ for which*

$$p(z_0) = q(\zeta_0) \quad \text{and} \quad p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U}),$$

and

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0).$$

Moreover

$$\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m \ \mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\}, \tag{1.2}$$

for some

$$m \geq n + \frac{|q'(0)| - \beta|z_0|^n}{|q'(0)| + \beta|z_0|^n}.$$

Lemma 1.3. [1] *Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist $z_0 \in \mathbb{U}$ such that $|\mathfrak{arg}\ p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\mathfrak{arg}\ p(z_0)| = \pi\alpha/2$ where $\alpha > 0$. Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = -i\alpha m\ , \qquad m \leq -\frac{1}{2}\left(a + \frac{1}{a}\right)\left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right)$$

when $\mathfrak{arg}\ p(z_0) = -\frac{\pi\alpha}{2}$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha m\ , \qquad m \geq \frac{1}{2}\left(a + \frac{1}{a}\right)\left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right)$$

when $\mathfrak{arg}\ p(z_0) = \frac{\pi\alpha}{2}$ where $p(z_0)^{\frac{1}{\alpha}} = \pm ia$ and $0 \leq \beta \leq 2\alpha$.

2. Main Results

Theorem 2.1. *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{U}$ with*

$$|\mathfrak{Im}\{C(z)\}| < \mathfrak{Re}\{B(z)\} \tag{2.1}$$

If $p(z) \in \mathcal{H}_\beta[1, n]$, $0 \leq \beta \leq 2$, and if

$$|\mathfrak{arg}\{B(z)zp'(z) + C(z)p(z)\}| < \frac{\pi}{2} + t(z), \tag{2.2}$$

where

$$t(z) = \begin{cases} \mathfrak{arg}\left\{B(z)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] + C(z)\right\} := \tau & \text{when} \quad \tau \in [0, \pi/2], \\ \\ \mathfrak{arg}\left\{B(z)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] + C(z)\right\} - \frac{\pi}{2} := \tau' & \text{when} \quad \tau' \in (\pi/2, \pi], \end{cases} \tag{2.3}$$

then we have

$$\mathfrak{Re}\{p(z)\} > 0. \quad (z \in \mathbb{U}) \tag{2.4}$$

Proof. By Lemma 1.3, if $\mathfrak{Re}\{p(z)\} > 0$ does not hold for all $z \in \mathbb{U}$, then there exists a point z_0, $|z_0| < 0$, such that

$$|\mathfrak{arg}\{p(z)\}| < \frac{\pi}{2} \quad for \quad (|z| < |z_0|) \quad and \quad |\mathfrak{arg}\{p(z_0)\}| = \frac{\pi}{2},$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = -ik$$

where

$$k \leq -\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] \quad when \quad \mathfrak{arg}\{p(z_0)\} = -\frac{\pi}{2},$$

and

$$k \geq \left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] \quad when \quad \mathfrak{arg}\{p(z_0)\} = \frac{\pi}{2},$$

where

$$p(z_0) = \pm ia \quad and \quad 0 \leq \beta \leq 2.$$

For the case $p(z_0) = ia, \ a > 0$, we have

$$\begin{aligned} |\mathfrak{arg}\{B(z_0)zp'(z_0) + C(z_0)p(z_0)\}| &= \left|\mathfrak{arg}\left\{p(z_0)\left[B(z_0)\frac{z_0p'(z_0)}{p(z_0)} + C(z_0)\right]\right\}\right| \\ &= \left|\mathfrak{arg}\left\{p(z_0)\left[B(z_0)ik + C(z_0)\right]\right\}\right|. \end{aligned} \tag{2.5}$$

By (2.1), we have $\mathfrak{Im}\{B(z_0)ik + C(z_0)\} > 0$. Therefore, from (2.5) we obtain

$$|\mathfrak{arg}\{B(z_0)zp'(z_0) + C(z_0)p(z_0)\}|$$

$$= \frac{\pi}{2} + \begin{cases} \mathfrak{arg}\{B(z_0)ik + C(z_0)\} & when \quad \mathfrak{arg}\{B(z_0)ik + C(z_0)\} \in [0, \pi/2], \\ \mathfrak{arg}\{B(z_0)ik + C(z_0)\} - \frac{\pi}{2} & when \quad \mathfrak{arg}\{B(z_0)ik + C(z_0)\} \in (\pi/2, \pi] \end{cases}$$

$$\geq \frac{\pi}{2} + \begin{cases} \mathfrak{arg}\left\{B(z_0)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] + C(z_0)\right\} := \tau & when \quad \tau \in [0, \pi/2], \\ \mathfrak{arg}\left\{B(z_0)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right] + C(z_0)\right\} - \frac{\pi}{2} := \tau' & when \quad \tau' \in (\pi/2, \pi] \end{cases}$$

$$= \frac{\pi}{2} + t(z_0).$$

This contradicts (2.2). For the case $p(z_0) = -ia, a > 0$, the proof runs as in the first case. □

Remark 2.2. Theorem 2.1 improves a result due to Miller and Mocanu [See [5], p. 208]. Also, it extends a result due to Nunokawa et al. [See [11], p. 3].

Corollary 2.3. *Let* $g(z) \in \mathcal{H}_\beta[1,n],\ 0 \le \beta \le 2$ *with*

$$|\mathfrak{Im}\left\{\frac{zg'(z)}{g(z)}\right\}| < 1,$$

and let $f \in \mathcal{A}_{n,b}$. *Suppose that*

$$|\mathfrak{Im}\{g(z)f'(z)\}| < \frac{\pi}{2} + \nu(z), \quad (z \in \mathbb{U})$$

where

$$\nu(z) = \begin{cases} \mathfrak{arg}\left\{i\left[\frac{2(n+1)+(\beta+b)(n-1)}{2+\beta+b}\right] + 1 - \frac{zg'(z)}{g(z)}\right\} := \lambda & \text{when} \quad \lambda \in [0, \pi/2], \\ \mathfrak{arg}\left\{i\left[\frac{2(n+1)+(\beta+b)(n-1)}{2+\beta+b}\right] + 1 - \frac{zg'(z)}{g(z)}\right\} - \frac{\pi}{2} := \lambda' & \text{when} \quad \lambda' \in (\pi/2, \pi]. \end{cases}$$

Then we have

$$\mathfrak{Re}\left\{\frac{g(z)f(z)}{z}\right\} > 0. \quad (z \in \mathbb{U})$$

Proof. We put $B(z) = 1, C(z) = 1 - zg'(z)/g(z)$ and $p(z) = g(z)f(z)/z$. Then $p(z) \in \mathcal{H}_{\beta+b}[1,n]$ and

$$|\mathfrak{Im}\{C(z)\}| < \mathfrak{Re}\{B(z)\} = 1. \quad (|z| < |z_0|)$$

Moreover, (2.2) becomes

$$|\mathfrak{arg}\{g(z)f'(z)\}| < \frac{\pi}{2} + \nu(z). \quad (z \in \mathbb{U})$$

Hence, applying Theorem 2.1, we obtain the desired result immediately. □

Remark 2.4. By taking $\beta + b = 2$ and $n = 1$, Corollary 2.3 reduces to a result obtained by Nunokawa et al. [See [11], p. 5]. Also, it improves a result due to Miller and Mocanu [See [5], p. 208]

Theorem 2.5. *Let* $B(z)$ *and* $C(z)$ *be analytic in* $\mathbb{U}$ *with* $B(z) \neq 0$. *Suppose that*

$$\mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\} \ge -T(n,\beta), \quad (z \in \mathbb{U}) \tag{2.6}$$

where

$$T(n,\beta) = \frac{n+1+\beta(n-1)}{1+\beta}, \tag{2.7}$$

for $0 \le \beta \le 1$ *and* $n \ge 1$. *If* $p(z) \in \mathcal{H}_\beta[0,n]$, *and if*

$$|B(z)zp'(z) + C(z)p(z)| < |B(z) + C(z)|, \quad (z \in \mathbb{U}) \tag{2.8}$$

then we have

$$|p(z)| < 1. \quad (z \in \mathbb{U})$$

Proof. By Lemma 1.2, if $p(z) \not\prec z$ in $\mathbb{U}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad and \quad \zeta_0, \quad |\zeta_0| = 1,$$

for which

$$p(z_0) = \zeta_0 \quad and \quad p(|z| < r_0) \subset \mathbb{U},$$

and

$$z_0p'(z_0) = m\zeta_0, \quad for \quad some \quad m \geq n + \frac{1-\beta}{1+\beta}.$$

We see that $m \geq T(n,\beta) \geq 1$. Then, by (2.8), we have

$$\begin{aligned}
|B(z_0)z_0p'(z_0) + C(z_0)p(z_0)| &= |mB(z_0) + C(z_0)| \\
&= |B(z_0)| \left| m + \frac{C(z_0)}{B(z_0)} \right| \\
&\geq |B(z_0)| \left| T(n,\beta) + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\} + i\mathfrak{Im}\left\{\frac{C(z_0)}{B(z_0)}\right\} \right| \\
&\geq |B(z_0)| \left| 1 + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\} + i\mathfrak{Im}\left\{\frac{C(z_0)}{B(z_0)}\right\} \right| \\
&= |B(z_0) + C(z_0)|.
\end{aligned}$$

which contradicts (2.8). Therefore, $|p(z)| < 1$ in $\mathbb{U}$. □

Theorem 2.6. *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{U}$ with $B(z) \neq 0$. Suppose that*

$$\mathfrak{Im}\left\{\frac{C(z)}{B(z)}\right\} \geq \frac{T(n,\beta)}{|B(z)|}, \quad (z \in \mathbb{U}) \tag{2.9}$$

where $T(n,\beta)$ is the same as (2.7) with $0 \leq \beta \leq 1$ and $n \geq 1$. If $p(z) \in \mathcal{H}_\beta[0,n]$, and if

$$|B(z)zp'(z) + C(z)p(z)| < \sqrt{1 + |B(z)|^2\left[\frac{zp'(z)}{p(z)} + \mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\}\right]^2}, \tag{2.10}$$

in $\mathbb{U}$, then we have

$$|p(z)| < 1. \quad (z \in \mathbb{U})$$

Proof. Applying the same method as in the proof of Theorem 2.2, if $p(z) \not\prec z$ in $\mathbb{U}$, then there exist points

$$z_0 = r_0e^{i\theta} \in \mathbb{U} \quad and \quad \zeta_0, \quad |\zeta_0| = 1,$$

for which

$$p(z_0) = \zeta_0 \quad and \quad p(|z| < r_0) \subset \mathbb{U},$$

and

$$\frac{z_0p'(z_0)}{p(z_0)} = m \quad for\ some \quad m \geq n + \frac{1-\beta}{1+\beta}.$$

Then we have

$$\begin{aligned}
|B(z_0)z_0p'(z_0) + C(z_0)p(z_0)| &= |p(z_0)| \left| B(z_0)\frac{z_0p'(z_0)}{p(z_0)} + C(z_0) \right| \\
&= |mB(z_0) + C(z_0)| \\
&= |B(z_0)| \left| m + \frac{C(z_0)}{B(z_0)} \right| \\
&= |B(z_0)| \left| m + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\} + i\mathfrak{Im}\left\{\frac{C(z_0)}{B(z_0)}\right\} \right|.
\end{aligned}$$

By (2.9), we have

$$|B(z_0)z_0p'(z_0)+C(z_0)p(z_0)| \geq |B(z_0)|\sqrt{\left[m+\mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right]^2+\frac{T(n,\beta)^2}{|B(z_0)|^2}}$$
$$\geq \sqrt{T(n,\beta)^2+|B(z_0)|^2\left[m+\mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right]^2}$$
$$\geq \sqrt{1+|B(z_0)|^2\left[\left|\frac{z_0p'(z_0)}{p(z_0)}\right|+\mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right]^2}$$

which contradicts (2.10). Therefore, $|p(z)|<1$ in $\mathbb{U}$. □

Remark 2.7. By taking $\beta=n=1$, Theorem 2.5 and Theorem 2.6 reduce to results obtained by Nunokawa et al. [See [11], p. 6]. Also, it improves a result due to Miller and Mocanu [See [5], p .206].

Theorem 2.8. *Let $p(z)\in\mathcal{H}_\beta[1,n]$ with $0\leq\beta\leq 1$ and $n\geq 1$ and*

$$\mathfrak{Re}\left\{2p(z)-\frac{zp''(z)}{p'(z)}-1\right\}>2\alpha,\quad (z\in\mathbb{U}) \tag{2.11}$$

Then we have

$$\mathfrak{Re}\{p(z)\}>\alpha,\quad (z\in\mathbb{U}) \tag{2.12}$$

where $\alpha<1$.

Proof. Let

$$q(z)=\frac{1-(2\alpha-1)z}{1-z},$$

where $0<\alpha<1$. We know that q is analytic and univalent in $\mathbb{U}$ with $q'(0)=2-2\alpha$ and $\mathfrak{Re}\{q(\mathbb{U})\}>\alpha$. So, $q\in\mathbf{Q}$ with $E(q)=1$. If (2.12) does not hold, means, $p(z)\not\prec q(z)$ in $\mathbb{U}$, then from lemma 1.2 there exists ζ_0 on $\partial\mathbb{U}$ with $p(z_0)=q(\zeta_0)$ such that $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$, for some

$$m\geq n+\frac{2-2\alpha-\beta}{2-2\alpha+\beta},$$

with $0\leq\beta\leq 2-2\alpha$ and $n\geq 1$ .We have $\mathfrak{Re}\{p(z_0)\}=\mathfrak{Re}\{q(\zeta_0)\}=\alpha$. Also, by (1.2) we have

$$\mathfrak{Re}\left\{1+\frac{z_0p''(z_0)}{p'(z_0)}\right\}\geq m\ \mathfrak{Re}\left\{1+\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}\right\}=m\ \mathfrak{Re}\left\{\frac{1+\zeta_0}{1-\zeta_0}\right\}=0.$$

Therefore, we have

$$\mathfrak{Re}\left\{2p(z_0)-\frac{z_0p''(z_0)}{p'(z_0)}-1\right\}=\mathfrak{Re}\left\{2q(\zeta_0)-\frac{z_0q''(z_0)}{q'(z_0)}-1\right\}\leq 2\mathfrak{Re}\{q(\zeta_0)\}=2\alpha,$$

which contradicts (2.11). This completes the proof. □

Remark 2.9. Theorem 2.8 improves a result due to Miller and Mocanu [See [5], p .207].

Theorem 2.10. *If $p \in \mathcal{H}_\beta[0,n]$ with $0 \leq \beta \leq 1$ and $n \geq 1$, then*

$$|zp'(z)| + \left|\frac{z^2p''(z)}{p(z)}\right| < \left[n + \frac{1-\beta}{1+\beta}\right]^2 \tag{2.13}$$

implies that $|p(z)| < 1$.

Proof. By Lemma 1.2, if $p(z) \not\prec q(z) = z$ in $\mathbb{U}$, then there exist points

$$z_0 = r_0e^{i\theta} \in \mathbb{U} \quad \textit{and} \quad \zeta_0, \quad |\zeta_0| = 1,$$

for which

$$p(z_0) = \zeta_0 \quad \textit{and} \quad p(|z| < r_0) \subset \mathbb{U},$$

and

$$\frac{z_0p'(z_0)}{p(z_0)} = m, \quad \textit{for some} \quad m \geq n + \frac{1-\beta}{1+\beta}.$$

By (1.2) we have

$$\mathfrak{Re}\left\{1 + \frac{z_0p''(z_0)}{p'(z_0)}\right\} \geq m\ \mathfrak{Re}\left\{1 + \frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}\right\} = m$$

Therefore, we have

$$\begin{aligned}
|z_0p'(z_0)| + \left|\frac{z_0^2p''(z_0)}{p(z_0)}\right| &= \left|\frac{z_0p'(z_0)}{p(z_0)}\right| \left[|p(z_0)| + \left|\frac{z_0p''(z_0)}{p'(z_0)}\right|\right] \\
&= \left|\frac{z_0p'(z_0)}{p(z_0)}\right| \left[1 + \left|\frac{z_0p''(z_0)}{p'(z_0)}\right|\right] \\
&\geq m\ \mathfrak{Re}\left\{1 + \frac{z_0p''(z_0)}{p'(z_0)}\right\} \\
&\geq m^2 \\
&\geq \left[n + \frac{1-\beta}{1+\beta}\right]^2.
\end{aligned}$$

which contradicts (2.13). Therefore, $|p(z)| < 1$ in $\mathbb{U}$. □

Remark 2.11. By taking $\beta = n = 1$, Theorem 2.10 reduces to a result due to Miller and Mocanu [See [5], p .207].

References

1. Amani, M., Aghalary, R. and Ebadian, A., Open-Door Lemma for Functions with Fixed Second Coefficients, Bull. Malays. Math. Sci. Soc. (2021).
2. Amani, M., Aghalary, R. and Ebadian, A., Extension of Nunokawa Lemma for Functions with Fixed Second Coefficient and Its Applications, Journal of Function Spaces, vol. 2021, Article ID 4564694, 11 pages, 2021.
3. Amani, M., Aghalary, R., Ebadian, A. and Cho, N. E., On extensions of the open door lemma, Mathematica Slovaca, vol. 72, no. 2, 2022, pp. 355-366.
4. Amani, M., Aghalary, R. and Ebadian, A., On meromorphic functions with fixed second coefficient, Asian-European Journal of Mathematics. Vol. 16, No. 01, 2350009 (2023).
5. Miller, S. S., Mocanu, P.T., Differential subordination and inequalities in the complex plane. J. Differ. Equ. 67(2),(1987):199-211.

6. Miller, S. S. and Mocanu, P. T., Differential subordinations and univalent functions, Michigan Math. J. 28 (1981), no. 2, 157172.
7. Miller, S. S. and Mocanu, P. T. and Reade, M. O., Bazilevic functions and generalized convexity, Rev. Roumaine Math. Pures Appl. 19 (1974), 213224.
8. Miller, S. S. and Mocanu, P. T., Differential subordinations. Theory and applications, Dekker, Inc. New York, (2000).
9. Nunokawa, M., "On the order of strongly starlikeness of strongly convex functions." Proceedings of the Japan Academy, Series A, Mathematical Sciences 69.7 (1993): 234-237.
10. Nunokawa, M., "On properties of non-Caratheodory functions." Proceedings of the Japan Academy, Series A, Mathematical Sciences 68.6 (1992): 152-153.
11. Nunokawa, M., Cho, N.E., Kwon, O.S., Sokol, J. On differential subordinations in the complex plane. J Inequal Appl 2015, 7 (2015).
12. Rosihan, M. A., Nagpal, S and Ravichandran, V., Second-order differential subordination for analytic functions with fixed initial coefficient, Bull. Malays. Math. Sci. Soc. (2)34 (2011), 611-629.