SOME RESULTS ON CONTINUOUS K-G-FRAMES IN HILBERT SPACES

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Abstract. In this paper, beside giving some results about continuous K-g-frame and showing the relationship between several types of its generalizations, we introduce frame operator for a continuous K-g-frame and then discuss some of its properties.

1. Introduction

Discrete frames for Hilbert spaces were formally defined by Duffin and Schaeffer [9] in 1952 for studying some deep problems in nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies et al. [7], and popularized from then on. After introducing the generalizations of frame such as continuous frame by Kaiser [11] and independently by Ali et al. [2, 3], and g-frame by Sun [13], Abdollahpour and Faroughi [1] and independently Dehghan and Hasankhani Fard [8], introduced the concept of continuous g-frame and investigated some of their properties.
Throughout this paper, $\mathcal{H}$ is a Hilbert space, I is a counting set, $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of Hilbert spaces, (Ω, μ) is a measure space with a positive

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measure μ, $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$ is a family of Hilbert spaces, $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$, $K \in B(\mathcal{H})$, and $R(K)$ is the range of the operator K.

Definition 1.1. [13] A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2.$$

Definition 1.2. [11] A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame for $\mathcal{H}$, if F is weakly measurable, i.e., for all $f \in \mathcal{H}$, $\omega \mapsto \langle f, F(\omega)\rangle$ is a measurable function on Ω, and there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

Definition 1.3. [1] Let (Ω, μ) be a measure space and let $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$ be a family of Hlibert spaces. Let $F \in \prod_{\omega\in\Omega} \mathcal{V}_\omega$. We say that F is strongly measurable if F is measurable as a mapping from Ω to $\bigoplus_{\omega\in\Omega} \mathcal{V}_\omega$, where

$$\prod_{\omega\in\Omega} \mathcal{V}_\omega = \Big\{ f : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{V}_\omega \,,\; f(\omega) \in \mathcal{V}_\omega \Big\}.$$

Definition 1.4. [1] We say that $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{V}_\omega) : \omega \in \Omega\}$ is a continuous generalized frame or simply a continuous g-frame with respect to $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ if
(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
(ii) there are two constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \int_\Omega \|\Lambda_\omega(f)\|^2 d\mu_\omega \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

L. Găvruta in [10] introduced K-frames in Hilbert spaces to study the atomic decomposition systems.

Definition 1.5. Let $K \in B(\mathcal{H})$. A sequence $\{f_i\}_{i\in I}$ is said to be a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

In [14] and [15], Y. Zhou and Y.C. Zhu studied K-g-frames in Hilbert spaces.

Definition 1.6. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

Rahimlou, et.al., introduced the concept of continuous K-frame in [12].

Definition 1.7. Let $K \in B(\mathcal{H})$ and $F : \Omega \to \mathcal{H}$ be weakly measurable. Then the map F is called a cK-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that for each $h \in \mathcal{H}$,

$$A\|K^*h\|^2 \leqslant \int_\Omega |\langle h, F(\omega)\rangle|^2 d\mu(\omega) \leqslant B\|h\|^2.$$

Recently, Alizadeh et.al in [4] introduced the concept of continuous K-g-frames in Hilbert spaces, and they studied some of their properties in [5].

Definition 1.8. A family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{V}_\omega)\,,\, \omega \in \Omega\}$ is called a continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$, if
(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable;
(ii) there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leqslant \int_\Omega \|\Lambda_\omega(f)\|^2 d\mu_\omega \leqslant B\|f\|^2, \;\; f \in \mathcal{H}.$$

The constants A and B are called the continuous K-g-frame bounds. Λ is called a tight continuous K-g-frame if

$$A\|K^*f\|^2 = \int_\Omega \|\Lambda_\omega(f)\|^2 d\mu_\omega, \;\; f \in \mathcal{H}.$$

2. Main results

In this section, we show some results about continuous K-g-frames. Then we introduce their frame operator.

Lemma 2.1. *A continuous K-frame is equivalent to a continuous K-g-frame whenever $\mathcal{V}_\omega = \mathbb{C}$, for all $\omega \in \Omega$.*

Lemma 2.2. *If $\Omega = \mathbb{N}$ and μ be a counting measure, then a continuous K-g-frame is a K-g-frame.*

Proposition 2.3. *If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$ with bound A, then $\{\Lambda_\omega K^*\}_{\omega\in\Omega}$ and $\{\Lambda_\omega K\}_{\omega\in\Omega}$ are tight continuous K-g-frame and tight continuous K^*-g-frame for $\mathcal{H}$ with respect to $\mathcal{V}_\omega$ with bound A, respectively.*

Proposition 2.4. *If $T \in B(\mathcal{H})$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame for $\mathcal{H}$, then $\{\Lambda_\omega T\}_{\omega\in\Omega}$ is a continuous T^*K-g-frame for $\mathcal{H}$.*

Lemma 2.5. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame with respect to $\{\mathcal{V}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with bounds A, B. Then for all $f, g \in \mathcal{H}$, the mapping*

$$\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$
$$\sigma(f, g) = \int_\Omega \langle f, \Lambda_\omega^*\Lambda_\omega(g)\rangle d\mu_\omega$$

is a bounded sesquilinear form and there exist a unique operator S such that for all $f, g \in \mathcal{H}$,

$$\langle Sf, f\rangle = \int_\Omega \|\Lambda_\omega f\|^2 d\mu_\omega.$$

The operator S is a bounded, linear and positive operator.

Definition 2.6. The operator S defined in Lemma 2.5 is called the continuous K-g-frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$. We show it by the notion

$$Sf = \int_\Omega \Lambda_\omega^* \Lambda_\omega f d\mu_\omega.$$

The synthesis and analysis operators of $\{\Lambda_\omega\}_{\omega\in\Omega}$ are defined as follow:

$$T : (\oplus_{\omega\in\Omega}\mathcal{V}_\omega, \mu)_{L_2} \to \mathcal{H},\ \langle TF, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu_\omega,\ \ F \in (\oplus_{\omega\in\Omega}\mathcal{V}_\omega, \mu)_{L_2},\ g \in \mathcal{H}$$

$$T^* : \mathcal{H} \to (\oplus_{\omega\in\Omega}\mathcal{V}_\omega, \mu)_{L_2},\ T^*(g)(\omega) = \Lambda_\omega g,\ g \in \mathcal{H},\ \omega \in \Omega.$$

Proposition 2.7. *Let $K \in B(\mathcal{H})$ and $R(K)$ be closed. Then $S : R(K) \to S(R(K))$ is a homeomorphism.*

Corollary 2.8. *The operator S is invertible on $R(K)$, if $R(K)$ be closed.*

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