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GENERAL VERSION OF THE SANDOR'S INEQUALITY

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Abstract. In this paper, we show Sandor type inequality for pseudo-integrals. Indeed, we state classic version of this inequality for pseudo-integrals. Some illustrate examples are given for theorems.

1. Introduction

The theory of fuzzy measures and fuzzy integral (Sugeno integral) has introduced by Sugeno [6] in his Ph.D. theses on 1974. From 2007, some authors have studied on some others fuzzy integral inequalities. Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Recently, Daraby et al. generalized Stolarsky, Hardy and Feng Qi type inequalities for pseudo-integrals ([2, 3, 4]).
Sandor's inequality in classical case is the following form.

Theorem 1.1. [1] *Let $f : [a, b] \to \mathbb{R}$ be a convex and non-negative function. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq \frac{1}{3}\left[f^2(a) + f(a)f(b) + f^2(b)\right], \tag{1.1}$$

holds.

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2. Preliminary

Now, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details.

Let $[a,b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b]\times[a,b] \to [a,b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a,b], \mathbf{0}\oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a,b]_+ = \{x|x \in [a,b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b]\times[a,b] \to [a,b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x\odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b], \mathbf{1}\odot x = x$.

We shall consider the semiring $([a,b],\oplus,\odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty)$, i.e., pseudo-operations are given with:

$$x \oplus y = g^{-1}(g(x)+g(x)) \qquad \text{and} \qquad x\odot y = g^{-1}(g(x)g(x)). \tag{2.1}$$

Then, the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the $g-$integral

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \tag{2.2}$$

The second class is when $x\oplus y = \max(x,y)$ and $x\odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f\odot dm = \sup_{x\in\mathbb{R}} \left(f(x)\odot\psi(x)\right),$$

where function ψ defines sup-measure m. We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = \text{ess sup}_\mu(x|x\in A) = \sup\left\{a|\mu(x|x \in A, x > a) > 0\right\}.$$

Theorem 2.3. *Let m be a* sup*-measure on $([0,\infty],\mathbb{B}[0,\infty])$, where $\mathbb{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = \text{ess sup}_\mu(\psi(x)|x \in A)$, and $\psi : [0,\infty] \to [0,\infty]$ is a continuous function. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0,\infty],\mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power $\lambda, \lambda \in (0,\infty)$) such that $\lim_{\lambda\to\infty} m_\lambda = m$.*

Theorem 2.4. *Let $([0,\infty],\sup,\odot)$ be a semiring , when $\odot$ is a generated with g, i.e., we have $x\odot y = g^{-1}(g(x)g(y))$ for every $x,y \in (0,\infty)$. Let*

m be the same as in Theorem 2.3, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0,\infty)$ such that for every continuous function $f : [0,\infty] \to [0,\infty]$,

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1} \left(\int g^\lambda(f(x))dx \right). \quad (2.3)$$

3. Main Results

In this section, we express Sandor's inequality for pseudo-integrals.

Theorem 3.1. *Let $f : [a,b] \to [c,d]$ be a continuous, convex and non-negative function and $g : [c,d] \to [0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)dx \right) \le \frac{1}{3} g\Big(\left[f_{\odot}^2(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^2(b) \right] \Big), \quad (3.1)$$

holds.

Corollary 3.2. *Let $f : [0,1] \to [c,d]$ be a continuous, convex and non-negative function and $g : [c,d] \to [0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[0,1]}^{\oplus} f_{\odot}^2(x)dx \right) \le \frac{1}{3} g \left[f_{\odot}^2(0) \oplus f(0) \odot f(1) \oplus f_{\odot}^2(1) \right], \quad (3.2)$$

holds.

Example 3.3. Let f and g are defined from $[0,1]$ to $[0,1]$ by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then we have

$$\frac{1}{4} = \frac{1}{1-0} \int_{[0,1]}^{\oplus} f_{\odot}^2(x)dx \le \frac{1}{3} g \left[f_{\odot}^2(0) \oplus f(0) \odot f(1) \oplus f_{\odot}^2(1) \right] = \frac{1}{3}.$$

We can not remove the assumption g is increasing in Theorem 3.1. The following example shows this fact.

Example 3.4. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Then we have

$$\frac{5}{6} = \frac{1}{1-0} \int_{[0,1]}^{\oplus} f_{\odot}^2(x)dx \not\le \frac{1}{3} g \left[f_{\odot}^2(0) \oplus f(0) \odot f(1) \oplus f_{\odot}^2(1) \right] = \frac{1}{3}.$$

Theorem 3.5. *Let $f : [a,b] \to [a,b]$ be a measurable comonotone function and $([a,b], \sup, \odot)$ be a simiring and m be the same as Theorems 2.3 and 2.4. If g is the continuous and increasing function, then the following inequality is holds*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[a,b]}^{\sup} f_{\odot}^2(x)dx \right) \le \frac{1}{3} g \left[f_{\odot}^2(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^2(b) \right], \quad (3.3)$$

holds.

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