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MOUNTAIN PASS SOLUTION FOR A $p(x)$-BIHARMONIC KIRCHHOFF TYPE EQUATION

MARYAM MIRZAPOUR

Department of Mathematics, Faculty of Mathematical Sciences, Farhangian University , Tehran, Iran
m.mirzapour@cfu.ac.ir

Abstract. In this paper we deal with the existence of weak solution for a $p(x)$-Kirchhoff type problem of the following form

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the Mountain Pass Theoem, we establish conditions ensuring the existence result.

1. Introduction

In this paper we study the following problem

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded smooth domain with smooth boundary $\partial\Omega$, $p(x) \in C(\overline{\Omega})$, $a, b > 0$ are constants, g is a continuous function, λ is a real parameter. Suppose that the nonlinearity $g(x,t) \in C(\overline{\Omega}, \mathbb{R})$ satisfies the following assumptions:

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($\mathbf{g_1}$) $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition and the subcritical growth condition, i.e. there exists a constant $c{\geq}0$ such that

$$|g(x,s)|\leq c_1(1+|s|^{q(x)-1}),$$

for all $(x,t)\in\Omega\times\mathbb{R}$ where $q(x)\in C_+(\overline{\Omega})$ and $q(x)<p_k^*(x)$.

($\mathbf{g_2}$) $g(x,s)=o(|s|^{p(x)-2}s)$ as $s\to 0$ uniformly with respect to $x\in\Omega$.

($\mathbf{g_3}$) There exist $M>0$ and $\theta\in\left(p^+,\dfrac{2(p^-)^2}{p^+}\right)$ such that $0<\theta G(x,s)\leq sg(x,s)$, for all $|s|\geq M$ and $x\in\Omega$ where $G(x,t)=\int_0^s g(x,\tau)\,d\tau$.

We mainly consider a new Kirchhoff problem involving the $p(x)$-biharmonic operator, that is, the form with a nonlocal coefficient $(a-b\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx)$. Its background is derived from nagative Young's modulus, when the atoms are pulled apart rather than compressed together and the strain is negative. Recently, the authors in [6] first studied this kind of problem

$$\begin{cases} -\left(a-b\int_\Omega|\nabla u|^2\,dx\right)\Delta u=\lambda|u|^{p-2}u & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}$$

where $2<p<2^*:=(2N)/(N-2)$, and they obtained the existence of solutions by using the mountain pass theorem. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem. We refer the readers to [1, 5, 7] and the references therein.

Now, we state our main result:

Theorem 1.1. *Assume that the function $q\in C(\overline{\Omega})$ satisfies*

$$1<p^-<p(x)<p^+<2p^-<q^-<q(x)<p_k^*(x):=\frac{Np(x)}{N-kp(x)}.$$

Then for any $\lambda\in\mathbb{R}$, with ($\mathbf{g_1}$)-($\mathbf{g_3}$) satisfied, problem (1.1) has a nontrivial weak solution.

2. Notations and preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega})=\{p(x);\ p(x)\in C(\overline{\Omega}),\ p(x)>1,\ \forall x\in\overline{\Omega}\}$, $p^+=\max\{p(x);\ x\in\overline{\Omega}\}$, $p^-=\min\{p(x);\ x\in\overline{\Omega}\}$;, $L^{p(x)}(\Omega)=\{u:\Omega\to\mathbb{R}$ measurable and $\int_\Omega|u(x)|^{p(x)}\,dx<\infty\}$, with the norm $|u|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\left\{\mu>0;\ \int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx\leq 1\right\}$.

Proposition 2.1 (See [3])**.** *The space $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e., $\frac{1}{p(x)}+\frac{1}{q(x)}=1$, for all $x\in\Omega$. For $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have $\left|\int_\Omega uv\,dx\right|\leq\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}\leq 2|u|_{p(x)}|v|_{q(x)}$.*

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as follows: $W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^{\alpha}u \in L^{p(x)}(\Omega), \ |\alpha| \leq k\}$, where $D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm $\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^{\alpha}u|_{p(x)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [3, 2].

Proposition 2.2 (See [3]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. If we replace $\leq$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\| = \inf\left\{\mu > 0 : \ \int_{\Omega} \left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$.

Remark 2.3. According to [4], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta\cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta\cdot|_{p(x)}$ are equivalent.

We consider the functional $\rho(u) = \int_{\Omega} |\Delta u|^{p(x)}\, dx$ and give the following fundamental proposition.

Proposition 2.4 (See [?]). *For $u \in X$ and $u_n \subset X$, we have*

(1) $\|u\| < 1(\textit{respectively} = 1; > 1) \Longleftrightarrow \rho(u) < 1(\textit{respectively} = 1; > 1)$;

(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) $\|u_n\| \to 0$ $(\textit{respectively} \to \infty) \Longleftrightarrow \rho(u_n) \to 0$ $(\textit{respectively} \to \infty)$.

3. Proof of Theorem 1.1

We say $u \in X$ is a weak solution of (1.1), if

$$(a - b\int_{\Omega} \frac{1}{p(x)})|\Delta u|^{p(x)}\, dx \int_{\Omega} |\Delta u|^{p(x)-2}\Delta u \Delta\varphi\, dx - \lambda\int_{\Omega} |u|^{p(x)-2}u\varphi\, dx = \int_{\Omega} g(x,u)\varphi\, dx,$$

where $\varphi \in X$. The energy functional $J : X \to \mathbb{R}$ associated with problem

$$\begin{aligned} J(u) = {} & a\int_{\Omega} \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx - \frac{b}{2}\left(\int_{\Omega} \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx\right)^2 \\ & - \lambda\int_{\Omega} \frac{1}{p(x)}|u|^{p(x)}\, dx - \int_{\Omega} G(x,u)\, dx \end{aligned} \tag{3.1}$$

for all $u \in X$ is well defind and of class C^1 in X. Moreover, we have

$$\langle J'(u),\varphi\rangle = (a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi\,dx$$
$$-\lambda\int_\Omega |u|^{p(x)-2}u\varphi\,dx - \int_\Omega g(x,u)\varphi\,dx. \tag{3.2}$$

Hence, we can observe that the critical points of J are weak solutions of problem (1.1).

Definition 3.1. Let $(X,\|\cdot\|)$ be a Banach space and $J \in C^1(X)$. We say that J satisfies the Palais-Smale condition at level c ($(PS)_c$ in short) if any sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^* as $n \to \infty$, has a convergent subsequence.

Lemma 3.2. *Assume that* $(\mathbf{g_1})$- $(\mathbf{g_3})$ *hold. Then the functional* J *satisfies the* $(PS)_c$ *condition, where* $c < \dfrac{a^2}{2b}$.

Lemma 3.3. *Assume that* g *satisfies* $(\mathbf{g_1})$- $(\mathbf{g_3})$. *Then* J *satisfies the Mountain Pass geometry, that is,*

(i) *there exists* $\rho,\alpha > 0$ *such that* $J(u) \geq \alpha > 0$, *for any* $u \in X$ *with* $\|u\| = \rho$.
(ii) *there exists* $e \in X$ *with* $\|e\| > \rho$ *such that* $J\ (e) < 0$.

By Lemmas 3.2, 3.3 and the fact that $J(0) = 0$, J satisfies the Mountain Pass Theorem. Therefor, problem (1.1) has indeed a nontrivial waek solution.

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