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ON A SOLUTION OF NONLOCAL $(p_1(x), p_2(x))$-BIHARMONIC PROBLEM

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Abstract. This article is concerned with the existence of weak solution for a class of elliptic Navier boundary value problem involving the $(p_1(x), p_2(x))$-biharmonic operator. By means of variational methods and the theory of variable exponent Sobolev spaces, we establish the existence of a non-trivial weak solution for problem.

1. Introduction

In recent years, a great attention has been paid to the study of various mathematical problems with variable exponent. Fourth order equations appears in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [5].
In this work, we consider the problem

$$\begin{cases} M_1\left(\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\,dx\right)\Delta^2_{p_1(x)}u+ \\ M_2\left(\int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\,dx\right)\Delta^2_{p_2(x)}u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded smooth domain with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p_i(x)} u := \Delta(|\Delta u|^{p_i(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator with ,$p_i(x) \in C(\overline{\Omega})$, $(i = 1, 2)$ such that $1 < p_i^- := \inf_{x\in\Omega} p_i(x) \leq p_i^+ := \sup_{x\in\Omega} p_i(x) < +\infty$, $M_i : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable functions and $f(x, u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.
Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [6]. Throughout this paper, we make the following assumptions on the function f and the Kirchhoff functions M_1 and M_2.

($\mathbf{H_0}$) $\exists\ m_0, m_1 > 0$ such that $M_1(t) \geq 0$ and $M_2(t) \geq 0$ for all $t \geq 0$.

($\mathbf{H_1}$) $\exists\ \mu_1, \mu_2 \in (0, 1)$ such that $\widehat{M_i}(t) \geq (1-\mu_i)M_i(t)t$ for all $t \geq 0$, where $\widehat{M_i}(t) = \int_0^t M_i(s)\, ds, i = 1, 2.$

($\mathbf{H_2}$) M_1, M_2 are differentiable and decreasing functions on $\mathbb{R}^+$.

($\mathbf{f_0}$) $|f(x, t)| \leq C(1 + |t|^{q(x)-1})$ for all $(x, t) \in \Omega \times \mathbb{R}$ with $C \geq 0$ and $1 < q(x) < p^*_M(x)$, where $p_M(x) = \max\{p_1(x), p_2(x)\}$, for all $x \in \overline{\Omega}$, and $p^*_M(x)$ is the critical exponent of p_M.

($\mathbf{f_1}$) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\frac{p_M^+}{1-\mu}}} = +\infty$, uniformly for a.e. $x \in \Omega$, where $\mu = \max\{\mu_1, \mu_2\}$ and $F(x, t) = \int_0^t f(x, s)\, ds.$

($\mathbf{f_2}$) There exists $\theta \geq 1$ such that $\theta G(x, t) \geq G(x, st)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$, where $G(x, t) = f(x, t)t - \frac{p_M^+}{1-\mu} F(x, t).$

($\mathbf{f_3}$) $\lim_{t\to 0} \frac{F(x,t)}{|t|^{p_M^+}} = 0$, uniformly for a.e. $x \in \Omega$.

Now, we are ready to state our main result:

Theorem 1.1. *Suppose that the conditions* ($\mathbf{H_0}$)-($\mathbf{H_2}$) *and* ($\mathbf{f_0}$)-($\mathbf{f_3}$) *hold true, then problem* (1.1) *has at least one nontrivial weak solution.*

2. Notations and preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\}$, $p^+ = \max\{p(x);\ x \in \overline{\Omega}\}$, $p^- = \min\{p(x);\ x \in \overline{\Omega}\}$;, $L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable and $\int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$, with the *Luxemburg* norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\}.$

Proposition 2.1 (See [3])**.** *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is separable, uniformly convex, reflexive and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $q(x)$ *is the conjugate function of* $p(x)$*, i.e.,* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$*, for all* $x \in \Omega$*. For* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$*, we have* $\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as $W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq k\}$, where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}} u,$

with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm $\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [2, 3]. For any $k \geq 1$, denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if} \quad kp(x) < N, \\ +\infty & \text{if} \quad kp(x) \geq N \end{cases}$$

Proposition 2.2 (See [3]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. If we replace $\leq$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $X = X_1 \bigcap X_2$ equipped with the norm $\|u\|_r = \|u\|_{p_1} + \|u\|_{p_2}$, where $X_i = \big(W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega)\big)$, $i = 1, 2$ and $\|u\|_r = \inf\{\mu > 0 : \int_\Omega \left|\frac{\Delta u(x)}{\mu}\right|^{r(x)} dx \leq 1\}$ equipped with the norm $\|u\| = \|u\|_{p_i}$.

Remark 2.3. According to [4], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta \cdot|_{p(x)}$ are equivalent.

We consider the functional $\rho(u) = \int_\Omega |\Delta u|^{p(x)} dx$ and give the following fundamental proposition.

Proposition 2.4 (See [1]). *For $u \in X$ and $\{u_n\} \subset X$, we have*

(1) $\|u\| < 1(\textit{respectively} = 1; > 1) \Longleftrightarrow \rho(u) < 1(\textit{respectively} = 1; > 1)$;
(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;
(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
(4) $\|u_n\| \to 0$ $(\textit{respectively} \to \infty) \Longleftrightarrow \rho(u_n) \to 0$ $(\textit{respectively} \to \infty)$.

Now, we recall the definition of the (C)-condition and then state a deformation lemma, which is fundamental in order to get some min-max theorems.

Definition 2.5. [7] let X be a Banach space and $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the Cerami c condition (we denote condition $(C)_c$), if

(**i**) any bounded sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence.
(**ii**) there exist constant $\delta, R, \beta > 0$ such that $\|J'(u)\|\|u\| \geq \beta$ for all $u \in J^{-1}([c - \delta, c + \delta])$ with $\|u\| \geq R$. If $J \in C^1(X, \mathbb{R})$ satisfies condition $(C)_c$ foe every $c \in \mathbb{R}$, we say that J satisfies condition (C).

Lemma 2.6. [8] *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$, be such that $\|e\| > r$ and $b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e)$. If*

J satisfies the condition (C_c) with $c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$, and $\Gamma := \gamma \in C([0,1],X)|\gamma(0) = 0, \gamma(1) = e$, then c is a critical value of J.

3. Proof of Theorem 1.1

Define $I_{p_1(x)} := \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\,dx$ and $I_{p_2(x)} := \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\,dx$. The Euler-Lagrange functional associated to (1.1) is

$$J(u) = \widehat{M_1}(I_{p_1(x)}) + \widehat{M_2}(I_{p_2(x)}) - \int_\Omega F(x,u)\,dx.$$

Moreover, the derivative of J is given by

$$\langle J'(u),v\rangle = M_1(I_{p_1(x)})\int_\Omega |\Delta u|^{p_1(x)-2}\Delta u\Delta v\,dx + M_2(I_{p_2(x)})\int_\Omega |\Delta u|^{p_2(x)-2}\Delta u\Delta v\,dx - \int_\Omega f(x,u)v\,dx,$$

for all $u, v \in X$. Then we know that the weak solution of (1.1) corresponds to the critical point of the functional J. These two Lemmas lead us to get the proof of our main result:

Lemma 3.1. *If $(\mathbf{H_0})$- $(\mathbf{H_2})$ and $(\mathbf{f_0})$- $(\mathbf{f_2})$ hold, then J satisfies the Cerami condition.*

Lemma 3.2. *If $(\mathbf{H_0})$- $(\mathbf{H_1})$ and $(\mathbf{f_1})$, $(\mathbf{f_3})$ hold true, then all the assertions in Lemma 2.6 are satisfied.*

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