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# RANGE OF IDEMPOTENT ADJOINTABLE OPERATORS ON HILBERT $C^*$-MODULES

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Abstract. Let $T$ and $S$ be idempotent adjointable operators on the Hilbert $C^*$-module $\mathcal{H}$ over a $C^*$-algebra $\mathcal{A}$. We establish that if there exist constants $\alpha_1, \alpha_2 > 0$ such that for all $x \in R(T)$ and $y \in \mathcal{R}(S)$

$$|x+y| \geq \alpha_1|x| \text{ and } |x+y| \geq \alpha_2|y|,$$

then $\mathcal{R}(T) \cap \mathcal{R}(S) = \{0\}$ and $\mathcal{R}(T) + \mathcal{R}(S)$ is orthogonality complemented submodule of $\mathcal{H}$. We also show that if $\Pi_1, \Pi_2$ are idempotents in $\mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi_1) \cap \mathcal{R}(\Pi_2) = \{0\}$ and $\mathcal{R}(\Pi_1) + \mathcal{R}(\Pi_2)$ is an orthogonally complemented submodule of $\mathcal{E}$, Then $\mathcal{R}(\Pi_1 + \Pi_2)$ is closed in $\mathcal{E}$ if and only if $\mathcal{R}(\Pi_1 - \Pi_2)$ is closed in $\mathcal{E}$.
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## 1. Introduction

Let $\mathcal{A}$ be a $C^*$-algebra. A pre-Hilbert $C^*$-module $\mathcal{H}$ over $\mathcal{A}$ is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ satisfying:

(1) $\langle x, x \rangle \geq 0,\ x \in \mathcal{X}$; $\langle x, x \rangle = 0$ if and only if $x = 0$.
(2) $\langle x, y \rangle^* = \langle y, x \rangle,\ x, y \in \mathcal{X}$.
(3) $\langle x, ya \rangle = \langle x, y \rangle a,\ x, y \in \mathcal{X},\ a \in \mathcal{A}$.

If the norm defined by $\|x\|^2 = \|\langle x, x \rangle\|$ for all $x \in \mathcal{X}$ is complete we say $\mathcal{X}$ is a Hilbert $C^*$-module. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert $C^*$-modules. Let

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$\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of all maps $T:\mathcal{H}\to\mathcal{K}$ for which there is an application $T^*:\mathcal{H}\to\mathcal{K}$ such that

$$\langle Tx,y\rangle=\langle x,T^*y\rangle,\quad x\in\mathcal{H},y\in\mathcal{K}. \tag{1.1}$$

With the abbreviation we denote by $\mathcal{L}(\mathcal{H},\mathcal{H})=\mathcal{L}(\mathcal{H})$. We denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and nullity of an operator $T$, respectively. Let $\mathcal{M}$ be a closed submodule of $\mathcal{X}$. Then we set

$$\mathcal{M}^\perp:=\{x\in\mathcal{X};\langle x,y\rangle=0,y\in\mathcal{M}\}\ .$$

We say that $\mathcal{M}$ is an orthogonally complemented submodule of $\mathcal{X}$ if $\mathcal{X}=\mathcal{M}+\mathcal{M}^\perp$. A closed submodule $\mathcal{M}$ is not necessarily orthogonally complemented. If $T\in\mathcal{L}(\mathcal{X})$ has closed range, it is known that $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are orthogonally complemented. The study of the properties of Hilbert $C^*$-modules and also the investigation of the facts that have been established in the Hilbert space and their generalization to the Hilbert $C^*$-module have been of interest to mathematical researchers, for examples see [3, 5].
For each idempotent operator $\Pi$,

$$\mathcal{R}(\Pi)\cap\mathcal{R}(I-\Pi)=0\quad\text{and}\quad\mathcal{R}(\Pi)+\mathcal{R}(I-\Pi)=\mathcal{E}. \tag{1.2}$$

A problem is that if $\Pi_1$ and $\Pi_2$ are idempotent operator, then do we have $\overline{\mathcal{R}(\Pi_1)+\mathcal{R}(\Pi_2)}$ is orthogonally complemented submodule? In the Hilbert space case we have the classic criteria of closeness for the sum of a couple of subspaces.

**Theorem 1.1.** [4, Propsitin 2.1],[1, Theorem 13] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed subspaces of $\mathcal{H}$. The following conditions are equivalent:*

(1) $\mathcal{H}_1+\mathcal{H}_2$ *is closed;*
(2) $\|P_1P_2-P_{\mathcal{H}_1\cap\mathcal{H}_2}\|<1$*;*
(3) $\mathcal{H}_1^\perp+\mathcal{H}_2^\perp$ *is closed;*
(4) $\mathcal{R}((I-P_1)P_2)$ *is closed;*
(5) $\mathcal{R}(I-P_1P_2)$ *is closed;*

## 2. Main Results

Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a Hilbert space $\mathcal{H}$. Recall that the cosine of angle between $\mathcal{M}$ and $\mathcal{N}$ defined as follows:

$$c_0(\mathcal{M},\mathcal{N}):=\sup\big\{\|\langle x,y\rangle\|:x\in\mathcal{M},y\in\mathcal{N},\|x\|\le1,\|y\|\le1\big\}.$$

We have the following characterization in Hilbert spaces:

**Theorem 2.1.** [1, Theorem 12] *The following statements are equivalent.*

(1) $c_0(\mathcal{M},\mathcal{N})<1$*;*
(2) $\mathcal{M}\cap\mathcal{N}=\{0\}$ *and* $\mathcal{M}+\mathcal{N}$ *is closed;*
(3) *There exist a constant* $\alpha>0$ *such that*

$$\|x+y\|\ge\alpha_1\|x\|\quad(x\in\mathcal{M},y\in\mathcal{N}). \tag{2.1}$$

In [2], it is defined the *separated pair* of the closed submodules of a Hilbert $C^*$-modules.

**Definition 2.2.** Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. Then we say that $(\mathcal{H},\mathcal{K})$ is a *separated pair* if

$$\mathcal{H}\cap\mathcal{K}=0 \text{ and } \mathcal{H}+\mathcal{K} \text{ is orthogonally complemented in } \mathcal{E}. \tag{2.2}$$

Now we give the following result.

**Theorem 2.3.** *Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. The following statements are equivalent:*

(i) *$(\mathcal{H},\mathcal{K})$ is a separated pair of orthogonally complemented submodules.*
(ii) *There are idempotents $\Pi_1$ and $\Pi_2$ in $\mathcal{L}(\mathcal{E})$ such that $\Pi_1\Pi_2=\Pi_2\Pi_1=0$, $\mathcal{R}(\Pi_1)=\mathcal{H}$ and $\mathcal{R}(\Pi_2)=\mathcal{K}$.*
(iii) *There is an idempotent $\Pi\in\mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi)=\mathcal{H}$ and $\mathcal{K}\subseteq\mathcal{N}(\Pi)$.*

**Corollary 2.4.** *Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. Then $(\mathcal{H},\mathcal{K})$ is a separated pair if and only if there exist constants $\alpha_1,\alpha_2>0$ such that $|x+y|\geq\alpha_1|x|$ and $|x+y|\geq\alpha_2|y|\quad(x\in\mathcal{H},y\in\mathcal{K})$.*

**Theorem 2.5.** *Let $\Pi_1,\Pi_2$ be idempotents in $\mathcal{L}(\mathcal{E})$ such that $(\mathcal{R}(\Pi_1),\mathcal{R}(\Pi_2))$ is a separated pair of orthogonally complemented submodules of $\mathcal{E}$. Then $\mathcal{R}(\Pi_1+\Pi_2)$ is closed in $\mathcal{E}$ if and only if $\mathcal{R}(\Pi_1-\Pi_2)$ is closed in $\mathcal{E}$.*

The following example shows that the separation condition in Theorem 2.5 is necessary.

**Example 2.6.** Let $\mathcal{K}$ be a separable Hilbert space and let $T$ be a non closed range operator on $\mathcal{K}$. Let $\mathcal{E}=\mathcal{K}\oplus\mathcal{K}$ and define idempotent operators $\Pi_1$ and $\Pi_2$ on $\mathcal{E}$ by

$$\Pi_1=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\quad \Pi_2=\begin{pmatrix} I & 0\\ T & 0\end{pmatrix}.$$

Note that $\mathcal{R}(\Pi_1)=\mathcal{K}\oplus 0$ and $\mathcal{R}(\Pi_2)=\{x\oplus Tx :\in\mathcal{K}\}$. Then $\mathcal{R}(\Pi_1)+\mathcal{R}(\Pi_2)=\mathcal{K}\oplus\mathcal{R}(T)$. This shows that $(\mathcal{R}(\Pi_1),\mathcal{R}(\Pi_2))$ is not separated pair of closed subspaces in $\mathcal{E}$. Since $\mathcal{R}(\Pi_1-\Pi_2)=0\oplus\mathcal{R}(T)$, so $\mathcal{R}(\Pi_1-\Pi_2)$ is not a closed subspace. Furthermore, the equation $\mathcal{R}(\Pi_1+\Pi_2)=\{2x\oplus Tx :\in\mathcal{K}\}$ yields that $\mathcal{R}(\Pi_1+\Pi_2)$ is a closed subspace in $\mathcal{E}$.

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