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HIGHER ORDER EXPONENTIALLY ISOMETRIC OPERATORS

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Abstract. For a positive integer m, a bounded linear operator T on a Hilbert space is called an exponentially m-isometric operator if $\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}e^{kT^*}e^{kT} = 0$. We show that for every non-empty compact subset K of pure imaginary axis, there exits an exponentially m-isometric operator T whose spectrum is K. Moreover, if $(T_n)_{n\geq 1}$ is a sequence of operators in this class that converges to T in the strong operator topology, then T is also an exponentially m-isometric operator.

1. Introduction

Throughout the paper, H stands for a Hilbert space and $B(H)$ denotes the space of all bounded linear operators on H. For a positive integer m, an operator $T \in B(H)$ is called an m-isometry if it satisfies the operator equation

$$\beta_m(T) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0$$

where T^* denotes the adjoint operator of T. Since the pioneer work of Agler [**?**], the study of m-isometries has become an active area of research

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in operator theory. Their applications to differential operator, disconjugacy and Brownian motion have been discussed in [?]. For more investigation on m-isometric operators one can see [?, ?].

An operator T is called an exponentially m-isometry if $\exp T$ is an m-isometric operator. Exponentially 1-isometric operators are simply exponentially isometries. The set of all exponentially m-isometric operators will be denoted by E_m. In [?] it has been proved that every m-isometry is an $(m+1)$-isometry and every invertible $2m$-isometry is a $(2m-1)$-isometry which implies that $E_{2m} = E_{2m-1}$.

Recall that $T \in B(H)$ is called an m-selfadjoint operator if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^{m-k} = 0$$

and T is skew-m-selfadjoint if iT is m-selfadjoint. These operators have been introduced and studied by Helton [?]. Moreover, for $m > 1$, the operator $T \in B(H)$ is said to be strict exponentially m-isometry if it is an exponentially m-isometric operator but not exponentially $(m-1)$-isometry. Similarly, one can define strict m-isometries and strict m-selfadjoint operators.

In [?, ?] authors investigate the sum of an m-isometric or an m-selfadjoint operator with a nilpotent operator and also the sum or product of two m-isometries or two m-selfadjoint operators. As an application of these results, we show that the sum of two commuting operators A and B that are, respectively, exponentially m-isometry and exponentially n-isometry is exponentially $(m+n-1)$-isometry. Also, we prove that if Q is a nilpotent operator of order l, $Q^l = 0$ and $Q^{l-1} \neq 0$, for some positive integer l, and A commutes with Q, then the sum $A + Q$ is an exponentially $(m + 2l - 2)$-isometric operator. It is known that the class of m-isometric and m-selfadjoint operators are stable under the powers [?, ?, ?]. We observe that the class of exponentially m-isometric operators is not stable under powers.

Also, we show that for each compact subset K of the pure imaginary line, there is an exponentially m-isometric operator T on a separable infinite Hilbert space whose spectrum is K. After that, we prove that limit of every sequence of exponentially m-isometric operators with respect to the strong operator topology is also an exponentially m-isometric operator. Furthermore, we show that every exponentially m-isometric diagonal, Toeplitz or multiplication operator is skew-m-selfadjoint. Moreover, we characterize normal, idempotent and weighted shift operators which are exponentially m-isometry.

2. MAIN RESULT

The skew-m-selfadjointness condition, $e^{-sT^*}e^{-sT} = \sum_{j=0}^{m-1} A_j s^j$ for each $s \in \mathbb{R}$ and some operators A_j, implies that the class of exponentially m-isometric operators contains all skew-m-selfadjoint operators. The following lemma implies that the class of skew-m-selfadjoint operators is a proper subset of E_m. In the following, $\langle .,. \rangle$ denotes the inner product on H. Moreover, for any vectors x and y in H, $x \otimes y$ denotes the rank one operator defined by

$$(x \otimes y)(z) = \langle z, y \rangle \, x.$$

Lemma 2.1. *Let $x, y \in H$. If $\langle x, y \rangle = 1$, then the following statements are equivalent:*
(a) $\|x\|\|y\| = 1$;
(b) *there exists a nonzero real number α such that $y = \alpha x$;*
(c) $\langle z, y \rangle \langle x, x \rangle y = \langle z, x \rangle \langle y, y \rangle x$, *for each $z \in H$;*
(d) $\langle x, z \rangle \langle z, y \rangle \geq 0$, *for each $z \in H$;*
(e) *$x \otimes y$ is selfadjoint.*

In the following example note that $x \otimes y$ is a nonzero idempotent if and only if $\langle x, y \rangle = 1$.

Example 2.2. Let H be an infinite-dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. For two distinct integers l and k greather that one, let $x = e_l$ and $y = e_l + e_k$. Then by Lemma **??**, $x \otimes y$ is an idempotent which is not selfadjoint. Moreover, let A be the unilateral weighted shift operator, $Ae_j = w_j e_{j+1}$, with weight $(w_j)_j$ on H such that $w_l = w_{l-1} = w_{k-1} = 0$, $\prod_{i=0}^{N-1} w_{i+j} = 0$ for all j and $N = \left[\frac{m+1}{2}\right]$. Since A and iA are unitarily equivalent, Proposition 2.5 of [**?**] implies that A is a skew-m-selfadjoint operator. Also, it is easily seen that the operator $x \otimes y$ commutes with A. Thus $A + 2\pi i x \otimes y$ is an exponentially m-isometric operator that is not skew-m-selfadjoint.

It is known that m-isometric and m-selfadjont operators are stable under powers [**?**, **?**, **?**]; meanwhile exponentially m-isometric operators are not. As an example, the operator $(iI)^n$ is exponentially isometry for all odd numbers n but it is not for any even number m. The sum of the commuting exponentially m-isometries as follows.

Theorem 2.3. *Let $A, B, Q \in B(H)$ be commuting operators. Suppose that $A \in E_m$, $B \in E_n$ and $Q^l = 0$ for some positive integer l. Then*
(i) *For each $k \in \mathbb{Z}$, $kA \in E_m$.*
(ii) *$A + B \in E_{m+n-1}$. In particular, for every pure imaginary number μ, $A + \mu I \in E_m$.*
(iii) *$A + Q \in E_{m+2l-2}$.*

Moreover, $A + Q$ is strict exponentially $(m + 2l - 2)$-isometry if and only if $Q^{*l-1}\beta_{m-1}(e^A)Q^{l-1} \neq 0$. In particular, for the case $m = 1$, $A + Q$ is strict exponentially $(2l - 1)$-isometry if and only if Q is nilpotent of order l.

Now, similar description for m-isometric operators [**?**], we will describe exponentially m-isometric operators with prescribed spectrum.

Theorem 2.4. *Let H be an infinite dimensional separable Hilbert space and $m > 1$ be an odd number. If K is a non-empty compact subset of pure imaginary axis, then there exists a strict exponentially m-isometric operator $T \in B(H)$ with spectrum K.*

Proposition 2.5. *Let T be an exponentially m-isometric operator. If one of the following statements holds, then T is skew-selfadjoint.*
(i) *T is a Toeplitz operator.*
(ii) *T is a diagonal operator.*
(iii) *$T = M_\varphi$ is the multiplication operator defined by $M_\varphi f = \varphi f$ on $L_2(\mu)$, for a σ-finite measure μ and a bounded Borel function φ, or on the Hardy space H^2 for $\varphi \in H^\infty$.*

Proposition 2.6. *Let T be an exponentially m-isometric operator. Then the following statements hold:*
(i) *If T is a normal operator then it is exponentially isometry.*
(ii) *If T is bounded below then it is invertible. Consequently, if T is an isometric operator, then it is unitary.*
(iii) *If T is an idempotent operator then $T = 0$.*

Suppose that $(T_n)_{n\geq 1}$ is a sequence of operators in E_m. If T_n converges to T then $T \in E_m$. Now, we consider the following question: If T_n converges to T in the strong operator topology, is $T \in E_m$? We will give positive answer to this question.

Proposition 2.7. *If $(T_n)_{n\geq 1}$ is a sequence of operators in E_m that converges to T in the strong operator topology, then $T \in E_m$.*

Corollary 2.8. *Suppose that $(T_n)_{n\geq 1}$ is a sequence of m-selfadjoint operators such that $T_n \to T$ in the strong operator topology. Then T is also an m-selfadjoint operator.*

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