NORM INEQUALITIES INVOLVING VARIOUS TYPES OF CONVEX FUNCTIONS

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Abstract. We present a general norm inequality for matrix functions, when various types of convexity are considered.

1. Introduction

Assume that $\mathcal{M}_n(\mathbb{C})$ is the C^*-algebra of all $n\times n$ complex matrices and $\mathcal{H}_n(\mathbb{C})$ is the real subspace of Hermitian matrices. A matrix $A\in\mathcal{H}_n(\mathbb{C})$ is said to be positive (semi-definite) and is denoted by $A\geq 0$ if all of its eigenvalues are non-negative. This induces a well-known partial order on $\mathcal{H}_n(\mathbb{C})$, the Löwner order:

$$A\leq B \iff B-A\geq 0 \qquad (A,B\in\mathcal{H}_n(\mathbb{C})).$$

A norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV|||=|||A|||$ holds for every $A\in\mathcal{M}_n(\mathbb{C})$ and all unitary matrices U,V. The most famous unitarily invariant norms on $\mathcal{M}_n(\mathbb{C})$ are the classes Schatten p-norms and Ky Fan k-norms, see [2].

For a real interval J, we denote by $\mathcal{H}_n(J)$ the set of all Hermitian matrices, whose eigenvalues are contained in J. Let f be a real function defined on an interval J. For any Hermitian matrix $A\in\mathcal{H}_n(J)$, the Hermitian matrix $f(A)$ is defined via the spectral decomposition of $A=U^*\mathrm{diag}(\lambda_1,\ldots,\lambda_n)U$ by $f(A)=U^*\mathrm{diag}(f(\lambda_1),\ldots,f(\lambda_n))U$, in which $\lambda_1,\ldots,\lambda_n$ are eigenvalues of A.

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Matrix means are known extensions of scalar means to the non-commutative setting. Let $A, B \geq 0$ and $t \in [0,1]$. The most famous matrix means are the arithmetic mean $A\nabla_t B = (1-t)A + tB$, the geometric mean $A\sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}$ and the Harmonic mean $A!_t B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$.

Recall that a real function $f : (0,\infty) \to (0,\infty)$ is called convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ holds for all $x, y \geq 0$ and every $t \in [0,1]$. It is known that [1] if f is a non-negative convex function on J, then

$$|||f(A\nabla_t B)||| \leq |||f(A)\nabla_t f(B)||| \tag{1.1}$$

holds for every unitarily invariant norm $|||\cdot|||$ and all $A, B \in \mathcal{H}_n(J)$.

We aim to generalize (1.1) for more type of convexity.

2. Main Result

The convex functions are defined by comparison of arithmetic means of points in the domain and in the image of a function. When the arithmetic function is replaced by other various means, some other types of convex functions are derived. Let α and β be two means. We say that a function f is α-β-convex when $f(\alpha(x,y)) \leq \beta(f(x), f(y))$ holds for all x, y in the domain of f. We refer the reader to [3, 5] for more information about various convexities and examples.

With respect to this notion, we present the next result, which provides a generalization of (1.1).

Theorem 2.1. *Let f be a positive function defined on $(0,\infty)$. If f is a α-β-convex function, then*

$$|||f(\alpha(A,B))||| \leq |||\beta(f(A), f(B))||| \tag{2.1}$$

holds for all positive matrices A, B. If $\alpha = \sharp_t$, then

$$\left|\left|\left|f\left(\exp\left(\frac{A+B}{2}\right)\right)\right|\right|\right| \leq |||\beta(f(\exp A), f(\exp B))||| \tag{2.2}$$

holds.

Theorem 2.1 enables us to examine many kinds of function rather that the classical convex functions. Following, we give some examples.

It is known that the function $x \mapsto \exp(x)$ and $x \mapsto x^r$ $(r < 0)$ are ∇-$\sharp$-convex, see [3, 5]. Theorem 2.1 then gives:

Corollary 2.2. *If $r < 0$, then*

$$|||(A+B)^r||| \leq 2^r\, |||A^r \sharp B^r||| \tag{2.3}$$

and

$$|||exp(1/2(A+B))||| \leq ||| \exp(A)\sharp \exp(B)||| \tag{2.4}$$

holds for all positive matrices A, B.

The function $f(x) = x^r$ is !-!-convex on $(0,\infty)$ for every $r \in [0,1]$. Applying Theorem 2.1 we have the next result.

Corollary 2.3. *If $r \in [0,1]$, then*

$$\left|\left|\left|\left(A^{-1}+B^{-1}\right)^{-r}\right|\right|\right| \leq 2^{1-r} \left|\left|\left|\left(A^{-r}+B^{-r}\right)^{-1}\right|\right|\right| \tag{2.5}$$

holds for all positive matrices A, B.

If $r \geq 0$ of $r \leq -1$, then $x \mapsto \exp(x^r)$ is a !-$\sharp$-convex function on $(0,\infty)$. Consequently, Theorem 2.1 implies that:

Corollary 2.4. *$f\, r \geq 0$ of $r \leq -1$, then*

$$\left|\left|\left|\exp\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-r}\right|\right|\right| \leq |||\exp(A^r)\sharp \exp(B^r)||| \tag{2.6}$$

holds for all positive matrices A, B.

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