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UNIFORM CONVERGENCE OF SEQUENCES OF COMPOSITION OPERATORS IN HARDY SPACE

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Abstract. We study the convergence of composition operators with respect to weak operator topology as well as strong operator topology on the Hardy space of analytic functions in the unit disk.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disk. For instance, $\mathcal{H}$ is the Hardy space H^2, or the Bergman space A^2. Given an analytic self-mapping φ on the unit disk, the *composition operator* $C_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known that the composition operator is bounded on the Hardy space H^2 and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

For a function $\psi \in \mathcal{H}$, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi.$$

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In the same manner we define the operator $D_\varphi : \mathcal{H} \to \mathcal{H}$ by

$$D_\varphi(f) = f' \circ \varphi.$$

According to [5, Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, the operator D_φ on H^2 is compact if and only if

$$\lim_{|z|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Now, let ψ be an analytic function on the unit disk, and define the *weighted composition-differentiation* operator $D_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ by the following relation:

$$D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi.$$

Our goal in this paper is to study the relationships between convergence of the sequence of operators D_{ψ_n,φ_n} in operator topologies from one hand, and the convergence of the sequences of functions ψ_n and φ_n on the other hand. G. Gunatillake's paper [3] studied the relationship between convergence of weighted composition operators C_{ψ_n,φ_n}, and the convergence of $\{\psi_n\}$ and $\{\varphi_n\}$. This result was extended by S. Mehrangiz and B. Khani-Robati [4] to generalized weighted composition operators on Bloch type spaces. Here we intend to generalize Gunatillake's result to weighted composition-differentiation operator $D_{\psi,\varphi}$ in the setting of classical Hardy spaces. More specifically, let $\mathcal{B}(H^2)$ denote the Banach algebra of all bounded linear operators on the Hilbert space H^2. It is rather well-known that the dual space of $\mathcal{B}(H^2)$ is too big, so that the weak and weak-star topology of this space is not so clear. For this reason, it is customary to equip this space with the weak operator topology, the strong operator topology, and the uniform operator topology. We intend to have a characterization of the convergence of D_{ψ_n,φ_n} to $D_{\psi,\varphi}$ with respect to operator topologies in terms of the convergence of $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the weak and strong operator topologies of H^2.

2. Preliminaries

Let f be an analytic function in the unit disk $\mathbb{D}$. The function f is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

It is easy to see that for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm of f in H^2 is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

It is well-known that for $f \in H^2$, the radial limit

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta}) = \lim_{r\to 1^-} f_r(e^{i\theta})$$

for almost every $\theta \in [0, 2\pi]$ exists. The function f^* is known as the radial function of f. The space H^2 is a functional Hilbert space, and its inner product is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}d\theta.$$

Since the evaluation functionals are bounded, the Hardy space is a reproducing kernel Hilbert space; this means that for each $w \in \mathbb{D}$, there is a function

$$K_w(z) = \frac{1}{1-\overline{w}z} \in H^2$$

such that every $f \in H^2$ has the following representation

$$f(w) = \langle f, K_w\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})\overline{K_w^*(e^{i\theta})}d\theta.$$

It is also well-known that the functional $w \mapsto f'(w)$ is bounded on H^2 ([**?**, Theorem 2.16]). It then follows from the Riesz representation theorem that there is a function $K_w^{(1)} \in H^2$ such that

$$f'(w) = \langle f, K_w^{(1)}\rangle, \quad f \in H^2.$$

It turns out that (see [2])

$$K_w^{(1)}(z) = \frac{z}{(1-\overline{w}z)^2}, \quad (z, w) \in \mathbb{D}\times\mathbb{D}.$$

In [1], we have proved the following theorems on the convergence in weak operator topology and strong operator topology.

Theorem 2.1. [1] *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2. Assume that each D_{ψ_n,φ_n} is bounded, and that $D_{\psi,\varphi}$ is a bounded nonzero operator on H^2. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in weak operator topology if and only if*
(a) ψ_n converges weakly to ψ in H^2,
(b) φ_n converges weakly to φ in H^2,
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

Theorem 2.2. [1] *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2 where ψ is nonzero. Assume that each D_{ψ_n,φ_n} and $D_{\psi,\varphi}$ are bounded operators on H^2 where $D_{\psi,\varphi}$ is nonzero. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology if and only if*
(a) ψ_n converges to ψ in H^2,
(b) φ_n converges to φ in H^2,
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

The following result whose proof will not be given here complements the above theorems. We use the notation $H_0^2 = \{f \in H^2 : f' \in H^2\}$.

Theorem 2.3. *Let* $\{\varphi_n\}_{n\geq 1}$ *and* φ *be analytic self-maps of the unit disk such that* $\sup \|\varphi_n\|_\infty < 1$*, and let* $\{\psi_n\}_{n\geq 1}$ *and* ψ *be elements in* H_0^2 *where* ψ *is nonzero and bounded. If* D_{ψ_n,φ_n} *converges to* $D_{\psi,\varphi}$ *in strong operator topology of* H_0^2*, then* D_{ψ_n,φ_n} *converges to* $D_{\psi,\varphi}$ *in uniform operator topology of* H_0^2*.*

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