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COMPLETE CONTINUITY OF WEIGHTED COMPOSITION-DIFFERENTIATION OPERATORS IN HARDY SPACE

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Abstract. In this paper we explore conditions under which every weighted composition-differentiation operator on the Hardy space $H^1(\mathbb{D})$ is completely continuous.

1. Introduction

Let $\mathcal{X}$ be a Banach space of analytic functions on the unit disk, and let φ be an analytic self-mapping on the unit disk. The *composition operator* $C_\varphi : \mathcal{X} \to \mathcal{X}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known that the composition operator is bounded on the Hardy space H^p and on the Bergman space A^p where p is a positive number. For a function $\psi \in \mathcal{X}$, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{X} \to \mathcal{X}$ is defined by

$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi.$$

Similarly, we can define the *composition-differentiation operator* $D_\varphi : \mathcal{X} \to \mathcal{X}$ by

$$D_\varphi(f) = f' \circ \varphi.$$

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In most cases the functional Banach space $\mathcal{X}$ equals either the Hardy space H^p or the Bergman space A^p. According to [4, Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, D_φ is compact on H^2 if and only if

$$\lim_{|z|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Now, let ψ be an analytic function on the unit disk, and define the *weighted composition-differentiation* operator $D_{\psi,\varphi} : \mathcal{X} \to \mathcal{X}$ by the following relation:

$$D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi.$$

This operator was recently studied in [1] and [3].

An operator $T : \mathcal{X} \to \mathcal{X}$ is said to be *completely continuous* if $x_n \to x$ weakly in $\mathcal{X}$, implies $\|Tx_n - Tx\| \to 0$. It is well-known that on a Banach space $\mathcal{X}$, every compact operator is completely continuous. On the other hand, if the Banach space $\mathcal{X}$ is reflexive, then completely continuous operators are compact. In this paper we shall focus on the non-reflexive Hardy space H^1, and try to find conditions under which the weighted composition-differentiation operator $D_{\psi,\varphi}$ is completely continuous. We shall provide characterizations for the complete continuity of this operator in terms of ψ and φ. More precisely, we prove that $D_{\psi,\varphi}$ is completely continuous if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.

2. Preliminaries

An analytic function f on the unit disk is said to belong to the Hardy space $H^p = H^p(\mathbb{D})$ if

$$\|f\|_{H^p}^p = \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For $1 \le p < \infty$, the Hardy space H^p is a Banach space of analytic functions, and for $p = 2$ it is a Hilbert space with the following inner product:

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})} d\theta,$$

where

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

is the boundary function of f. It is easy to see that for $f \in H^2$ with Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$, the norm of f is given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^\infty |a_n|^2.$$

Recall that an operator $T : \mathcal{X} \to \mathcal{X}$ is compact if for every bounded sequence (x_n) in $\mathcal{X}$, the sequence (Tx_n) has a convergent subsequence. We remark that for $1 < p < \infty$, the Hardy space H^p is reflexive; meaning that it is isometrically isomorphic with its dual. We know that on reflexive Banach spaces, an operator T is compact if and only if it is completely continuous. In this paper, we concentrate on the non-reflexive Banach space H^1 and the composition-differentiation operator $D_{\psi,\varphi}$ on H^1. We will find conditions on the function φ to ensure that the operator $D_{\psi,\varphi}$ is completely continuous on H^1.

3. Main Result

In the following theorem we shall characterize the complete continuity of composition-differentiation operator in terms of ψ and φ.

Theorem 3.1. *Let $\psi \in H^1$ and φ be a self-map on $\mathbb{D}$. Assume that $D_{\psi,\varphi}$ is bounded on H^1. Then $D_{\psi,\varphi}$ is completely continuous on H^1 if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.*

Proof. Let $D_{\psi,\varphi}$ be completely continuous, and let $\mathbb{T}$ denote the unit circle. Assume that $f \in L^\infty(\mathbb{T})$ and let $\hat{f}(n)$ be its n-th Fourier coefficient. By Riemann-Lebesgue lemma we have

$$\int_{\mathbb{T}} f(z)\overline{z}^n \mathrm{d}m = \hat{f}(n) \to 0, \quad n \to \infty.$$

This means that $\{z^n\}$ converges to zero weakly in $L^1(\mathbb{T})$, and hence weakly in H^1. Since $D_{\psi,\varphi}$ is completely continuous, it follows that

$$\|D_{\psi,\varphi}(z^n)\|_{H^1} \to 0, \quad n \to \infty.$$

On the other hand, for each $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} |\psi| \mathrm{d}m &\le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi| \mathrm{d}m \\ &= \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi||\varphi|^{n-1} \mathrm{d}m \\ &\le \int_{\mathbb{T}} n|\psi||\varphi|^{n-1} \mathrm{d}m \\ &= \|D_{\psi,\varphi}(z^n)\|_{H^1} \to 0, \quad n \to \infty. \end{aligned}$$

Therefore the integral on the left-hand side must be zero, from which it follows that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.

Conversely, Let (f_n) be a weak null sequence in H^1. It follows that $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. Using this fact together with the assumption that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$, we conclude that

$$D_{\psi,\varphi}(f_n)(e^{i\theta}) = \psi(e^{i\theta}) f_n'(\varphi(e^{i\theta})) \to 0, \quad a.e. \,\mathrm{in}\, \mathbb{T}.$$

It now follows that $D_{\psi,\varphi}(f_n)$ converges to zero in measure in $L^1(\mathbb{T})$ (see [5, page 74]). Moreover, the boundedness of $D_{\psi,\varphi}$ on H^1 implies that $D_{\psi,\varphi}(f_n) \to 0$ in the weak topology of H^1, and hence in the weak topology of $L^1(\mathbb{T})$. Finally, we invoke the fact that weak convergence of a given sequence together with its convergence in measure implies its norm convergence (see [2, page 295]), that is, $\|D_{\psi,\varphi}(f_n)\|_{H^1} \to 0$ as $n \to \infty$. □

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