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BOUNDEDNESS OF COMPOSITION OPERATORS IN POLYDISK

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Abstract. In this paper we investigate conditions on the symbol function to guarantee that the composition operator from the Bergman space of the polydisk to the Bergman space of the unit disk is bounded.

1. introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ is the space of analytic functions f in $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < +\infty,$$

where

$$dA_\alpha(z) = \pi^{-1}(\alpha+1)(1-|z|^2)^\alpha dx\, dy$$

is the weighted area measure in the unit disk. It is well-known that $A^2_\alpha(\mathbb{D})$ equipped with the inner product

$$\langle f, g\rangle = (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha dA(z),$$

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is a Hilbert space with the following reproducing kernel (see [4])

$$K_w(z) = \frac{1}{(1 - z\overline{w})^{\alpha+2}}.$$

We mean by polydisk the subset $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ of the n-dimensional complex space. Now let $\mathrm{Hol}(\mathbb{D}^n)$ denote the space of holomorphic functions on $\mathbb{D}^n$. The weighted Bergman space on the polydisk $\mathbb{D}^n$ is defined by

$$A^2_\alpha(\mathbb{D}^n) = \mathrm{Hol}(\mathbb{D}^n) \cap L^2(\mathbb{D}^n, dV_\alpha)$$

where

$$dV_\alpha(z) = dA_\alpha(z_1) \cdots dA_\alpha(z_n),$$

and

$$dA_\alpha(z_k) = \pi^{-1}(\alpha + 1)(1 - |z_k|^2)^\alpha dx_k dy_k, \quad 1 \leq k \leq n.$$

This means that a function $f(z_1, ..., z_n)$ in $\mathrm{Hol}(\mathbb{D}^n)$ belongs to $A^2_\alpha(\mathbb{D}^n)$ if

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} |f(z_1, ..., z_n)|^2 dA_\alpha(z_1) \cdots dA_\alpha(z_n) < +\infty.$$

The reproducing kernel of $A^2_\alpha(\mathbb{D}^n)$ is given by (see the papers [5] and [6])

$$K_z(w) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{z_j} w_j)^{\alpha+2}} = K_{z_1}(w_1) \cdots K_{z_n}(w_n).$$

Let $\Phi : \mathbb{D}^m \to \mathbb{D}^n$ be a holomorphic mapping (m, n are positive integers):

$$\Phi(z) = \big(\varphi_1(z), \ldots, \varphi_n(z)\big), \quad z = (z_1, \ldots, z_m) \in \mathbb{D}^m.$$

Consider the composition operator

$$C_\Phi : A^2_\alpha(\mathbb{D}^n) \to A^2_\beta(\mathbb{D}^m),$$

defined by $C_\Phi(f) = f \circ \Phi$. Moreover, if $\psi : \mathbb{D}^n \to \mathbb{C}$ is holomorphic, then the weighted composition operator $C_{\psi,\Phi}$ is defined by

$$C_{\psi,\Phi}(f) = \psi \cdot f \circ \Phi, \quad f \in A^2_\alpha(\mathbb{D}^n).$$

In this paper, we shall focus on the composition operator

$$C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D}).$$

This problem can then be studied for $C_\Phi : A^2_\alpha(\mathbb{D}^k) \to A^2_\alpha(\mathbb{D})$, and our choice $k = 2$ is for simplicity. We shall prove that if φ and ψ are analytic self mappings of the unit disk, and $\Phi = (\varphi, \psi) : \mathbb{D} \to \mathbb{D}^2$ is a holomorphic function such that $\|\varphi\psi\|_\infty < 1$, then $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded. We should mention that this problem for the Hardy space is already known; see [3]. For recent work on this topic see the papers [1] and [2].

2. Preliminaries

Let $F(z,w) \in A^2_\alpha(\mathbb{D}^2)$. Note that

$$C_\Phi F(z) = F(\varphi(z), \psi(z)).$$

Let

$$F(z,w) = \sum_{n=0}^{\infty} z^n F_n(w) = \sum_{n=0}^{\infty} w^n G_n(z).$$

Since for $m \neq n$ we have

$$\int_{\mathbb{D}^2} z^n \overline{z}^m F_n(w)\overline{F_m(w)} dA_\alpha(z) dA_\alpha(w) = 0,$$

it follows that

$$\begin{aligned} \|F\|^2_{A^2_\alpha(\mathbb{D}^2)} &= \sum_{n=0}^{\infty} \|z^n\|^2_{A^2_\alpha(\mathbb{D})} \|F_n\|^2_{A^2_\alpha(\mathbb{D})} \\ &= \sum_{n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)} \|F_n\|^2_{A^2_\alpha(\mathbb{D})}. \end{aligned}$$

Similarly, we see that

$$\|F\|^2_{A^2_\alpha(\mathbb{D}^2)} = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)} \|G_n\|^2_{A^2_\alpha(\mathbb{D})}.$$

Now let σ be a number satisfying

$$\|\varphi\psi\|_\infty = \sup_{z\in\mathbb{D}} |\varphi(z)\psi(z)| < \sigma < 1.$$

Then we can find measurable disjoint subsets Ω_1 and Ω_2 in the unit disk such that $\int_{\Omega_1\cup\Omega_2} dA_\alpha(z) = 1$, and $|\varphi(z)| < \sqrt{\sigma}$, a.e. in Ω_1, and $|\psi(z)| < \sqrt{\sigma}$, a.e. in Ω_2. To see this we define

$$\Omega_1 = \{z : |\varphi(z)| < \sqrt{\sigma},\ a.e.\},$$

and

$$\Omega_2 = \{z : |\varphi(z)| \geq \sqrt{\sigma},\ a.e.\}.$$

Clearly if $z \notin \Omega_1$, then $z \in \Omega_2$ and $|\varphi(z)| \geq \sqrt{\sigma}$. Hence we must have $|\psi(z)| < \sqrt{\sigma}$ since otherwise we have $|\varphi(z)\psi(z)| \geq \sigma$ which is not possible. This argument will be used in the proof of the main result.

3. Main result

We begin by proving that if $\Phi = (\varphi, \psi)$ is a holomorphic mapping from the unit disk to $\mathbb{D}^2$, then the composition operator $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded.

Theorem 3.1. *Let $\Phi = (\varphi, \psi)$ where φ and ψ are analytic self-mappings of the unit disk satisfying $\|\varphi\psi\|_\infty < 1$. Then $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded.*

Sketch of proof. It is clear that

$$\|C_\Phi F\|^2_{A^2_\alpha(\mathbb{D})} = \int_{\Omega_1} |C_\Phi F|^2 + \int_{\Omega_2} |C_\Phi F|^2.$$

Then we approximate

$$\int_{\Omega_1} |C_\Phi F|^2$$

by σ, norm of C_ψ and norm of F in the Bergman space $A^2_\alpha(\mathbb{D}^2)$. Similarly, one approximates

$$\int_{\Omega_2} |C_\Phi F|^2$$

by σ, norm of C_φ, and norm of F in the Bergman space $A^2_\alpha(\mathbb{D}^2)$. Finally,

$$\|C_\Phi F\|^2_{A^2_\alpha(\mathbb{D})} \le C_\sigma \Big(\|C_\varphi\|^2 + \|C_\psi\|^2 \Big) \|F\|_{A^2_\alpha(\mathbb{D}^2)}.$$

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