The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

REMARKS ON THE TRUNCATED WALLIS PRODUCT

MEHDI HASSANI*

Mehdi Hassani, Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran
mehdi.hassani@znu.ac.ir

Abstract. In this paper we study the truncated Wallis product, by showing that for each fixed integer $m \geqslant 1$, there exists computable constants $C'_1, \ldots, C'_m$, such that as $n \to \infty$,

$$\prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)} = \left(1 + \sum_{k=1}^{m} \frac{C'_k}{n^k}\right) \frac{\pi}{2} + O\left(\frac{1}{n^{m+1}}\right).$$

1. Introduction

The Wallis product for π obtained in 1655 by John Wallis and appeared one year later in his *Arithmetica Infinitorum* [13, p. 179] in the following form

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times 11 \times 11 \times 13 \times 13 \times \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times 10 \times 12 \times 12 \times 14 \times \cdots}.$$

See [9], [10, Chap. 3] and [11] for a detailed description of Wallis' work. In modern terminology and notation, the Wallis product for π reads as follows

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k \cdot 2k}{(2k-1)(2k+1)}. \tag{1.1}$$

2020 *Mathematics Subject Classification. Primary 40A20; Secondary 41A60, 33B15*
Key words and phrases. Wallis product, asymptotic expansion, Gamma and beta functions.

* Speaker.

Several researchers have established interesting properties of (1.1), including new proofs, generalizations, inequalities and connection with the probability theory. See [1, 4, 5, 8, 12, 14] and the references given there.

Standard proofs of the Wallis product for π runs over the integration of the powers of sine or cosine functions (see for example [6, Sec. 9.18]). Letting for each positive integer n,

$$I_n := \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x,$$

integration by parts gives $I_n = \frac{n-1}{n} I_{n-2}$. Repeated using this recurrence relation, we deduce that

$$I_{2n} = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}, \quad \text{and} \quad I_{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1}.$$

Dividing I_{2n+1} by I_{2n} we get $\frac{\pi}{2} = \mathcal{W}_n \eta_n$, where $\mathcal{W}_n$ is the truncated Wallis product given by

$$\mathcal{W}_n = \prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)}, \tag{1.2}$$

and

$$\eta_n = \frac{I_{2n}}{I_{2n+1}}. \tag{1.3}$$

Since $0 \leqslant \sin x \leqslant 1$ for $0 \leqslant x \leqslant \frac{\pi}{2}$, we observe that the sequence $(I_n)_{n \geqslant 1}$ is strictly decreasing, and consequently $1 \leqslant \eta_n \leqslant 1 + \frac{1}{2n}$. Thus, $\eta_n \to 1$ as $n \to \infty$. Equivalently, $\mathcal{W}_n \to \frac{\pi}{2}$ as $n \to \infty$, implying (1.1).

In this note we are motivated by studying the truncated form of the Wallis product. Considering the notion of asymptotic series [3, Sec. 1.5] due to Poincaré, we obtain an asymptotic series for the factor η_n, as follows.

Theorem 1.1. *Let $m \geqslant 1$ be fixed integer. There exists computable constants $C_1, \ldots, C_m$ such that as $n \to \infty$,*

$$\eta_n = 1 + \sum_{k=1}^{m} \frac{C_k}{n^k} + O\left(\frac{1}{n^{m+1}}\right). \tag{1.4}$$

Therefore, we have

$$\frac{\pi}{2} = \left(1 + \sum_{k=1}^{m} \frac{C_k}{n^k}\right) \mathcal{W}_n + O\left(\frac{1}{n^{m+1}}\right). \tag{1.5}$$

Remark 1.2. The value of the coefficients C_k are given by

$$C_k = \sum_{j=0}^{k} G_j\left(\frac{3}{2}, 1\right) G_{k-j}\left(\frac{1}{2}, 1\right), \tag{1.6}$$

where

$$G_k(a,b) = \binom{a-b}{k} B_k^{(a-b+1)}(a), \tag{1.7}$$

with $B_n^{(\ell)}(x)$ denoting the generalized Bernoulli polynomials, defined for integers $\ell \geqslant 0$ by

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\ell} \mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(\ell)}(x)}{n!} t^n, \qquad |t| < 2\pi.$$

By computation, we have

$$\eta_n = P\left(\frac{1}{n}\right) + O\left(\frac{1}{n^{11}}\right),$$

where

$$P(t) = 1 + \frac{1}{4}t - \frac{3}{32}t^2 + \frac{3}{128}t^3 + \frac{3}{2048}t^4 - \frac{33}{8192}t^5 - \frac{39}{65536}t^6 + \frac{699}{262144}t^7 + \frac{4323}{8388608}t^8 - \frac{120453}{33554432}t^9 - \frac{208749}{268435456}t^{10}.$$

Corollary 1.3. *Let $m \geqslant 1$ be fixed integer, $\mathcal{W}_n$ defined by (1.2), and the constants $C'_1, \ldots, C'_m$ defined by the recurrence*

$$\sum_{j=0}^{k} C_j C'_{k-j} = 0, \tag{1.8}$$

with the initial values $C_0 = C'_0 = 1$ and $C_1, \ldots, C_m$ given in (1.6). Then, as $n \to \infty$ we have

$$\mathcal{W}_n = \left(1 + \sum_{k=1}^{m} \frac{C'_k}{n^k}\right)\frac{\pi}{2} + O\left(\frac{1}{n^{m+1}}\right).$$

Remark 1.4. By computation, we have

$$\mathcal{W}_n = Q\left(\frac{1}{n}\right)\frac{\pi}{2} + O\left(\frac{1}{n^{11}}\right),$$

where

$$Q(t) = 1 - \frac{1}{4}t + \frac{5}{32}t^2 - \frac{11}{128}t^3 + \frac{83}{2048}t^4 - \frac{143}{8192}t^5 + \frac{625}{65536}t^6 - \frac{1843}{262144}t^7 + \frac{24323}{8388608}t^8 + \frac{61477}{33554432}t^9 - \frac{14165}{268435456}t^{10}.$$

2. Proofs

Proof of Theorem 1.1. The idea to obtain an asymptotic series for the factor η_n is relating it by the Euler gamma function [7, Eq. 5.2.1], which is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t,$$

and by analytic continuation for $\Re(z) \leqslant 0$ with simple poles of residue $\frac{(-1)^n}{n!}$ at $z = -n$, with $n \in \mathbb{N}$. To make this connection, we use the notion of the Beta function $\mathrm{B}(a, b)$ [7, Eq. 5.12.1], which is defined for complex variables a and b with $\Re(a) > 0$ and $\Re(b) > 0$ as follows

$$\mathrm{B}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, \mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.1}$$

The following trigonometric integral representation [7, Eqs. 5.12.2] holds the Beta function

$$\int_0^{\frac{\pi}{2}} \sin^{2a-1} x \cos^{2b-1} x \, \mathrm{d}x = \frac{1}{2} B(a, b).$$

Here we let $a = \frac{z+1}{2}$ with $\Re(z) > -1$, and $b = \frac{1}{2}$. By using (2.1) we deduce that

$$\int_0^{\frac{\pi}{2}} \sin^z x \, \mathrm{d}x = \frac{\Gamma(\frac{1}{2})}{2} \frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}.$$

We recall that the Wallis product (1.1) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [7, Eq. 5.4.6] are the same [2]. Hence, for each complex number z with $\Re(z) > -1$, we get

$$\int_0^{\frac{\pi}{2}} \sin^z x \, \mathrm{d}x = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}.$$

By using this identity, and recalling (1.3), we obtain the following Γ-representation for η_n,

$$\eta_n = \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)^2}.$$

Asymptotic expansion for the ratio of two gamma functions [7, Eqs. 5.11.13, 5.11.17, 24.16.1] asserts that for any complex constants a and b, if $z \to \infty$ in the sector $|\arg(z)| \leqslant \pi - \delta < \pi$, then

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{k=0}^{\infty} \frac{G_k(a, b)}{z^k},$$

where $G_k(a, b)$ is defined in (1.7). Considering the notion of asymptotic series [3, Sec. 1.5], due to Poincaré, we may read the above asymptotic, for any fixed integer $m \geqslant 1$, as the following truncated form

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \left(\sum_{k=0}^{m} \frac{G_k(a, b)}{z^k} + O\left(\frac{1}{|z|^{m+1}}\right)\right). \tag{2.2}$$

By using (2.2) we obtain

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)} = n^{\frac{1}{2}} \left(\sum_{k=0}^{m} \frac{G_k(\frac{3}{2}, 1)}{z^k} + O\left(\frac{1}{n^{m+1}}\right)\right),$$

and

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = n^{-\frac{1}{2}}\left(\sum_{k=0}^{m}\frac{G_k(\frac{1}{2},1)}{z^k} + O\left(\frac{1}{n^{m+1}}\right)\right).$$

Note that $G_0(a,b) = 1$ [7, Eqs. 5.11.15]. Thus, multiplying the above expansions gives (1.4) with C_k as in (1.6). This completes the proof. □

Proof of Corollary 1.3. By using the relation (1.5) we deduce that

$$\mathcal{W}_n = \left(1+\sum_{k=1}^{m}\frac{C_k}{n^k}\right)^{-1}\left(\frac{\pi}{2}+O\left(\frac{1}{n^{m+1}}\right)\right)$$
$$= \left(1+\sum_{k=1}^{m}\frac{C_k}{n^k}\right)^{-1}\frac{\pi}{2}+O\left(\frac{1}{n^{m+1}}\right).$$

We consider the Taylor expansion of the function $t \mapsto (1+t)^{-1}$ as $t \to 0$, and we let $t = \sum_{k=1}^{m}\frac{C_k}{n^k}$, where as assumed $n \to \infty$. Thus, there exists the constants $C'_1, \ldots, C'_m$ such that

$$\left(1+\sum_{k=1}^{m}\frac{C_k}{n^k}\right)^{-1} = 1+\sum_{k=1}^{m}\frac{C'_k}{n^k}+O\left(\frac{1}{n^{m+1}}\right),$$

or equivalently

$$\left(1+\sum_{k=1}^{m}\frac{C_k}{n^k}\right)\left(1+\sum_{k=1}^{m}\frac{C'_k}{n^k}\right) = 1+O\left(\frac{1}{n^{m+1}}\right).$$

Comparing the coefficients of the both sides, we observe that the recurrence (1.8) holds for each k with $1 \leqslant k \leqslant m$. This completes the proof. □

References

1. J. T. Chu, A modified Wallis product and some applications, *Amer. Math. Monthly* **69** (1962), 402–404.
2. Ó. Ciaurri, E. Fernández, V. Emilio and L. Juan, The Wallis product and $\Gamma(1/2) = \sqrt{\pi}$ are the same. (Spanish), *Gac. R. Soc. Mat. Esp.* **21** (2018), 566.
3. N. G. De Bruijn, *Asymptotic methods in analysis*, North-Holland Publishing Co., Amsterdam, 1961.
4. C. J. Everett, Inequalities for the Wallis product, *Math. Mag.* **43** (1970), 30–33.
5. P. Levrie and W. Daems, Evaluating the probability integral using Wallis's product formula for π, *Amer. Math. Monthly* **116** (2009), 538–541.
6. S. M. Nikolsky, *A course of mathematical analysis*, Vol. 1, Mir Publishers, Moscow, 1977.
7. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
8. T. J. Osler, Morphing Lord Brouncker's continued fraction for π into the product of Wallis, *Math. Gaz.* **95** (2011), 17–22.
9. T. J. Osler, The tables of John Wallis and the discovery of his product π, *Math. Gaz.* **94** (2010), 430–437.
10. R. Roy, *Series and products in the development of mathematics*, Vol. 1, Second edition, Cambridge University Press, Cambridge, 2021.

11. J. F. Scott, *The Mathematical Work of John Wallis, D.D., F.R.S., (1616-1703)*, Chelsea Publishing Company, 1981.
12. L. Short, Some generalizations of the Wallis product, *Internat. J. Math. Ed. Sci. Tech.* **23** (1992), 695–707 .
13. J. Wallis, *Arithmetica Infinitorum*, 1656.
14. J. Wästlund, An elementary proof of the Wallis product formula for pi, *Amer. Math. Monthly* **114** (2007), 914–917.