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ON $\mathcal{MT}$-CYCLIC CONTRACTIONS

HOSEIN LAKZIAN*

Department of Mathematics, Payame Noor University, P.O. Box 19395-4697, Tehran, I.R. of Iran
h_lakzian@pnu.ac.ir

Abstract. In this paper, we introduced the notion Hardy-Rogers $\mathcal{MT}$-cyclic contraction. Using this concept, we investigate the existence of best proximity point for such mappings in metric spaces. The uniqueness of this point will be obtain by imposing an additional condition, so called "property UC". At the end, using the definition of $\mathcal{MT}$-cyclic orbital contraction, we shall prove and discuss the existence and uniqueness of fixed point of such mappings in the setting of metric space and b-metric space.

1. Introduction

In the last decades, both fixed point theory and best proximity point theory have been appreciated by several authors, see e.g. [1-6]. In this paper, we examine existence and uniqueness of best proximity points and fixed points for generalized $\mathcal{MT}$- cyclic contractions and $\mathcal{MT}$-cyclic orbital contractions with respect to φ in the context of metric space. Let A and B be nonempty subsets of metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a *cyclic* if $T(A) \subset B$ and $T(B) \subset A$, see e.g. [6] and [12]. For any nonempty subsets A and B of X, we let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

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* Speaker.

A point $x \in A \cup B$ is called to be a best proximity point for T if $d(x, Tx) = dist(A, B)$. Note that if $A = B$ then the best proximity point of T turns into fixed point of T.

The concept of *property UC* was introduced by Suzuki *et al.* [14] as follows:

A pair (A, B) is said to satisfy the property UC if the following holds:

(UC) If $\{(x_n)_{n=1}^{\infty}\}$ and $\{(x'_n)_{n=1}^{\infty}\}$ are sequences in A and $\{y_n\}_{n=1}^{\infty}$ is a sequence in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim\limits_{n\to\infty} d(x'_n, y_n) = d(A, B)$ then $\lim\limits_{n\to\infty} d(x_n, x'_n) = 0$.

Definition 1.1. [8] A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-*function* if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup\limits_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$).

Remark 1.2. [8] It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example 1.3. [8] Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t} & , \text{if } t \in (0, \frac{\pi}{2}] \\ 0 & , \text{otherwise.} \end{cases}$$

Since $\limsup\limits_{s\to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

The aim of this paper is generalization of Theorem 1 in [10] by applying the notion of $\mathcal{MT}$-cyclic contraction with respect to a $\mathcal{MT}$-function φ.For convenience of the reader, we recall some of $\mathcal{MT}$-cyclic contractions in the framework of complete metric spaces which are defined in some papers:
For mapping $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$; T is called [6] [*$\mathcal{MT}$-cyclic contraction*] if

$$d(Tx, Ty) \le \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))dist(A, B);$$

[13][*$\mathcal{MT}$-cyclic Kannan contraction*] if

$$d(Tx, Ty) \le \frac{1}{2}\varphi(d(x, y))\Big(d(x, Tx) + d(y, Ty)\Big) + (1 - \varphi(d(x, y)))dist(A, B);$$

[4][*$\mathcal{MT}$-cyclic Reich contraction*] if

$$d(Tx, Ty) \le \frac{1}{3}\varphi(d(x, y))\Big(d(x, y)+d(x, Tx)+d(y, Ty)\Big)+(1-\varphi(d(x, y)))dist(A, B);$$

[2][*generalized $\mathcal{MT}$-cyclic contraction*] if

$$d(Tx, Ty) \le \varphi(d(x, y)) \max\{d(x, y), d(x, Tx), d(y, Ty)\}+(1-\varphi(d(x, y)))dist(A, B).$$

It is showed there exists an example give a map T which is a $\mathcal{MT}$-cyclic contraction but not a cyclic contraction; see Example A in [6].

2. Best Proximity point for Hardy-Rogers $\mathcal{MT}$-cyclic contraction

In this section, we present our main results. We, first, introduce the *generalized $\mathcal{MT}$-cyclic contraction* with respect to auxiliary $\mathcal{MT}$-function φ.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(HRMT1) $T(A) \subset B$ and $T(B) \subset A$;

(HRMT2) there exists a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$\begin{aligned} d(Tx, Ty) \leq &\frac{\varphi(d(x, y))}{5}\Big(d(x, y) + d(x, Tx) + d(Ty, y) + d(x, Ty) + d(Tx, y)\Big) \\ &+ \big(1 - \varphi(d(x, y))\big) dist(A, B), \end{aligned}$$

for all $x \in A$ and $y \in B$. Then T is called a *Hardy-Rogers $\mathcal{MT}$-cyclic contraction with respect to φ* on $A \cup B$.

In what follows that we establish the following theorem for best proximity point which is one of the main results in this paper.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X, d) and (A, B) satisfies the property UC. Let $T : A \cup B \to A \cup B$ be a cyclic map and let φ be a $\mathcal{MT}$-function. Suppose that A is complete and T is a Hardy-Rogers $\mathcal{MT}$-cyclic contraction with respect to φ. Then the following hold:*

(i) *T has a best proximity point z in A.*
(ii) *z is a unique fixed point of T^2 in A.*
(iii) *$\{T^{2n}x\}$ converges to z for every $x \in A$.*
(iv) *T has at least one best proximity point in B.*
(v) *If (B, A) satisfies the property UC, then Tz is unique best proximity point in B and $\{T^{2n}y\}$ converges to Tz for every $y \in B$.*

3. Best Proximity point for $\mathcal{MT}$-cyclic orbital contraction in b-metric spaces

In this section we obtain fixed point theorem for $\mathcal{MT}$-cyclic orbital contraction in b-metric spaces.

Bakhtin [3] and Czerwik [5] introduced b-metric spaces (a generalization of metric spaces) and proved the contraction principle in this framework.

Definition 3.1. [3] and [5] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s), is called a b-metric space with coefficient s.

We first introduce the concept of MT-cyclic orbital contractions.

Definition 3.2. Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MTO1) $T(A) \subset B$ and $T(B) \subset A$;

(MTO2) for some $x \in A$ there exists a MT-function $\varphi_x : [0, \infty) \to [0, 1)$ such that

$$d(T^{2n}x, Ty) \leq \varphi_x(d(T^{2n-1}x, y))d(T^{2n-1}x, y) \quad \text{for any } y \in A \text{ and } n \in \mathbb{N}.$$

Then T is called a *$\mathcal{MT}$-cyclic orbital contraction with respect to φ_x* on $A \cup B$.

The following example give a map T which is a $\mathcal{MT}$-cyclic orbital contraction but not a cyclic orbital contraction.

We obtain a unique fixed point for such a map as follows.

Theorem 3.3. *Let A and B be nonempty closed subsets of a complete b-metric space (X, d, s) and $T : A \cup B \to A \cup B$ be a $\mathcal{MT}$-cyclic orbital contraction with respect to φ. Then $A \cap B$ is nonempty and T has a unique fixed point.*

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