GENERALIZED HERMITE-HADAMARD INEQUALITY ON SEMISPHERE

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Abstract. In this paper we investigate the notions of $P-$convex and strongly $P-$convex functions defined on convex subsets of unit semisphere $\mathbb{R}^3$. Some versions of Hermite-Hadamard inequality are given in this setting.

1. Introduction

The Hermite-Hadamard inequality for a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

$I \subset \mathbb{R}$, has received renewed attention by many authors [3]. Many particular cases in several variables have been investigated by S.S. Dragomir in [5, 6]. Some improvements of 1.1 are studied in [4, 8]. The study of convex set and functions in semisphere, has several more accurate results and applications (see [9, 10]). Let us recall some of notions and results from differential geometry often used in what follows, see [1, 7] and references therein. A subset S of the unit sphere $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$, called convex if any two points x, $y \in S$ can be joined by a unique minimizing geodesic whose image belongs to S. Let S be a nonempty convex subset of

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S^2. A function $f : S \to \mathbb{R}$ is said to be quasiconvex if for every x, $y \in S$ and every $t \in [0, 1]$,

$$f(\gamma(t)) \leq \max\{f(x), f(y)\}, \tag{1.2}$$

where $\gamma : [0, 1] \to S$ is the unique minimal geodesic in S with $\gamma(0) = x$ and $\gamma(1) = y$.

For every $p, q \in S^2$ with $\omega := \arccos\langle p, q\rangle = d(p, q) < \pi($ d is called the intrinsic distance or Riemannian distance on S^2), the unique minimal geodesic in S^2 joining p and q is given by the following formula

$$\gamma(t) = \frac{\sin((1-t)\omega)}{\sin\omega}p + \frac{\sin(t\omega)}{\sin\omega}q,\ \ t \in [0, 1]. \tag{1.3}$$

Let $M \subseteq \mathbb{R}^3$ be a 2-surface and $f : M \to \mathbb{R}$ be an integrable function. If $F : D \to M$ is a C^1 parametrization of M, where D is an open subset of $\mathbb{R}^2$ in $uv-$plane then, the surface integral of f on $R := F(D) \subseteq M$ is defined by

$$\int_R f ds := \int\int_D f(F(u, v))\|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\|dudv.$$

Note that, the surface integral does not depend on parametrization. Recall the following result from [2].

Lemma 1.1. *Let $0 < \omega_0 < \pi$. Then, for every $0 \leq \theta < 2\pi$ the curve*

$$\alpha_\theta(\varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),\ \ \varphi \in [0, \omega_0],$$

is the unique minimal geodesic from $\tilde{p} := (0, 0, 1)$ to

$$q := (\sin\omega_0\cos\theta, \sin\omega_0\sin\theta, \cos\omega_0).$$

The Hermite-Hadamard inequality for a convex function on semisphere is investigated in [2]. Our goal in this paper is to establish an analogue of the Hermite-Hadamard inequality for $P-$convex and strongly $P-$convex functions defined on semisphere of S^2.

2. $P-$convexity and Hermite-Hadamard inequality

In this section the Hermite-Hadamard inequality for $P-$convex and strongly convex functions on hemispheres is considered.

Definition 2.1. Let S be a nonempty convex subset of S^2 and $f : S \to \mathbb{R}^+$ be a real valued function, $\mathbb{R}^+ := [0, +\infty]$. Then,
(i) f is said to be $P-$convex (or belong to the class $P(I)$) if it is nonnegative and for every $x, y \in S$ and every $t \in [0, 1]$,

$$f(\gamma(t)) \leq f(x) + f(y), \tag{2.1}$$

(ii) f is said to be strongly $P-$convex if it is nonnegative and there exists a constant $c > 0$ such that for every $x, y \in S$ and every $t \in (0, 1)$,

$$f(\gamma(t)) \leq f(x) + f(y) - ct(1-t)d^2(x, y), \tag{2.2}$$

where $\gamma : [0, 1] \to S$ is the unique minimal geodesic in S with $\gamma(0) = x$ and $\gamma(1) = y$.

It is easy to see that $P(I)$ contain all non-negative convex and quasiconvex functions defined on proper convex subsets of sphere are $P-$convex. In the following example we introduce a $P-$convex function defined on a convex subset of S^2 which is not quasiconvex.

Example 2.2. Define the non-negative function $f : C \to \mathbb{R}$ as

$$f(x) := 2\varphi_0^2 - d^2(x, \tilde{p}),$$

where $C := B(\bar{p}, \varphi_0)$, $0 < \varphi_0 < \pi/2$. Then, f is not a quasiconvex on C.

Now we are in a position to establish the Hermite-Hadamard inequality for $P-$convex functions defined on the semispheres of S^2.

Theorem 2.3. *Let that $f : C \to \mathbb{R}$ be a $P-$convex integrable function. Then, the following inequalitiy holds*

$$f(\tilde{p}) \leq \frac{2}{area(C)} \int_C f ds \leq 2f(\tilde{p}) + \frac{1}{\pi \sin \varphi_0} \int_{\partial C} f(\sigma(\tau)) d\tau, \tag{2.3}$$

where σ is the parametrization of ∂C by arc length and $C := B(\bar{p}, \varphi_0)$.

Next result is an improvement of lemma 2.2 in[2] for strongly $P-$convex functions.

Theorem 2.4. *Let S be a convex subset of S^2 and $q \in S^2$. Suppose that $f : S \to \mathbb{R}$ is a real valued function. Then, f is strongly $P-$convex on S with constant $\lambda > 0$ if and only if for every $x \in S$ the function $z \mapsto f(z) - \lambda d^2(z, x)$ is $P-$convex on S.*

The following establish a version of Hermite-Hadamard inequality for strongly $P-$convex functions.

Theorem 2.5. *Let that $f : C \to \mathbb{R}$ be a strongly $P-$convex integrable function with constant $\lambda > 0$. Then, the following inequalitiy holds*

$$g(\tilde{p}) \leq \frac{2}{area(C)} \int_C g ds \leq 2g(\tilde{p}) + \frac{1}{\pi \sin \varphi_0} \int_{\partial C} g(\sigma(\tau)) d\tau, \tag{2.4}$$

where σ is the parametrization of ∂C by arc length, $g(z) := f(z) - \lambda d^2(z, x)$ and $C := B(\bar{p}, \varphi_0)$..

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