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FIXED POINT APPROXIMATIONS FOR MAPPINGS IN GEODESIC SPACES

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Abstract. In this paper, we study fixed point approximations for mappings in geodesic spaces, and we prove some stability results in fixed point theory for contraction mappings in geodesic spaces. we also consider some extention theorems for these spaces.[1]

1. Introduction

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subseteq R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(x1, x2, x3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.

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A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Definition 1.1. Let X be a $CAT(0)$ space and $A \subseteq X$. We say X is convex, if for all $x, y \in A$ we have $[x, y] \subseteq X$.

Definition 1.2. let X be a $CAT(0)$ space and $A \subseteq X$. A is called geodesically bounded if A does not contain a geodesic ray.

we give some basic properties of metric segments in $CAT(0)$ spaces.

Remark 1.3. ([6]) Let X be a $CAT(0)$ space and let $x, y \in X$ such that $x \neq y$ and $s, t \in [0, 1]$. Then $(1-t)x \bigoplus ty = (1-s)x \bigoplus sy$ if and only if $s = t$.

Lemma 1.4. *([6]) Let X be a $CAT(0)$ space and let $x, y \in X$ such that $x \neq y$. then*

1. $[x, y] = \{(1-t)x \oplus ty | t \in [0, 1]\}$.
2. *$d(x, z) + d(z, y) = d(x, y)$ if and only if $z \in [x, y]$.*
3. *The mapping $f : [0, 1] \longrightarrow [x, y], f(t) = (1-t)x \oplus ty$ is continuous and bijective.*

Lemma 1.5. *([6]) Let X be a $CAT(0)$ space. then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 1.6. *([1]) Let (X, d) be a $CAT(0)$ space, $p, q, x, y \in X$ and $t \in [0, 1]$. Then*

$$d((1-t)p \oplus tq, (1-t)x \oplus ty) \leq \max\{d(p, x), d(q, y)\}.$$

Fixed point theorems in $CAT(0)$ spaces have been developed in several recent papers including [2, 3, 4, 5]. The existence of fixed points for nonexpansive mappings in a complete $CAT(0)$ space was proved by Kirk ([2]) as follows:

Theorem 1.7. *Suppose K is a nonempty, bounded, closed and convex subset of complete $CAT(0)$space and suppose $f : K \longrightarrow K$ is nonexpansive. Then f has a fixed point.*

Theorem 1.8. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Suppose $f : K \longrightarrow X$ is nonexpansive mapping for which*

$$\inf\{d(x, f(x)) : x \in K\} = 0$$

Then f has a fixed point in K.

2. Main results

Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X, d). Consider

$$A := \{A : A \text{ is a contraction selfmap on } K \text{ with } Fix(A) \neq \emptyset\},$$

with

$$\rho(A, B) = \sup\{d(Ax, Bx) : x \in K\}$$

for every $A, B \in A$. Then (A, ρ) is a complete metric space.

Definition 2.1. Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X, d). A map $A : K \to K$ is said to have the stable fixed point property if there exist $x \in Fix(A)$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni (B \in A, \rho(A, B) \leq \delta) \Longrightarrow$$
$$\exists z \in K \ni (Bz = z, d(z, x) \leq \varepsilon).$$

Theorem 2.2. *Let $K \subseteq X$ be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space (X, d), $x \in X$, and $A \in A$. Soppose $\lambda : X \to [0, 1]$ be a continuous map. Define B on K by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Then B is contraction and $Bz \in K$ for all $z \in K$.*

Proof. Since K is a closed convex subset of a complete $CAT(0)$ space, $Bz \in K$ for all $z \in K$. To see that B is contraction, Let $z_1, z_2 \in K$, Then by definition of B ,we have

$$\begin{gathered}
d(\lambda(z_1)x \oplus (1 - \lambda(z_1))Az_1, \lambda(z_2)x \oplus (1 - \lambda(z_2))Az_2) \\
\leq d(\lambda(z_1)x \oplus (1 - \lambda(z_1))Az_1, \lambda(z_1)x \oplus (1 - \lambda(z_2))Az_2) \\
+d(\lambda(z_1)x \oplus (1 - \lambda(z_2))Az_2, \lambda(z_2)x \oplus (1 - \lambda(z_2))Az_2) \\
\leq |\lambda(z_1) - \lambda(z_2)|d(Az_1, Az_2) + |\lambda(z_1) - \lambda(z_2)|d(x, x) \\
\leq |\lambda(z_1) - \lambda(z_2)|\alpha d(z_1, z_2).\square
\end{gathered}$$

Theorem 2.3. *Let $A \in A$ and $\varepsilon > 0$. then there exists $\delta > 0$ such that for each $B \in A$ satisfying $\rho(A, B) \leq \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in Fix(A)$ such that $d(x, y) \leq \varepsilon$.*

Proof. In contrary, suppose,

$$\exists \varepsilon > 0 \quad \forall n \quad B_n \in A \quad \textit{such that} \quad \rho(A, B_n) \leq \frac{1}{n}$$

and

$$\exists x_n \in K \quad \textit{such that} \quad B_n x_n = x_n$$

and $d(x_n, y) > \varepsilon$ for all $y \in Fix(A)$. Since K is compact, we may assume without loss of generality that there exists $x \in K$ such that $x_n \to x$ as $k \to \infty$. So

$$\begin{aligned}
d(Ax, x) &\leq d(Ax, Ax_n) + d(Ax_n, B_n x_n) + d(B_n x_n, x_n) + d(x_n, x) \\
&\leq d(Ax, Ax_n) + \frac{1}{n} + d(x_n, x).
\end{aligned}$$

Therefore $Ax = x$ hence $x \in Fix(A)$ and $d(x_n, x) > \varepsilon$ for all n. This is a contradiction.□

In view of this result, It is natural to ask,does for each $A \in A$, A have the stable fixed point property?

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall B \in A$ satisfying $\rho(A, B) \leq \delta$, $\exists y \in Fix(A)$ such that $By = y$, and $d(x, y) \leq \varepsilon$.

Example 2.4. Put $X := \mathbb{R}$, $K := [0, 1]$ and $Ax = x$ for all $x \in K$ so $Fix(A) = K$. and let $A_n x = (1 - \frac{1}{n})x$ and $B_n x = \min\{x + \frac{1}{n}, 1\}$ for all n. Therefore $A_n, B_n \to A$ and $Fix(A_n) = \{0\}$ and $Fix(B_n) = [1 - \frac{1}{n}, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this paper that for a typical $A \in A$ the answer is positive.

Theorem 2.5. *Let (X, d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Then there exist $B \in A$ and $\delta > 0$ such that $\rho(A, B) \leq \varepsilon$ and $Bz = x$ for each $z \in K$ satisfying $d(z, x) \leq \delta$.*

Proof. By continuity of A there exists $\delta > 0$ such that $d(z, x) \leq \delta$ implies that $d(Az, x) \leq \varepsilon$ for all $z \in K$. Since (X, d) is normal, by Urysohn's theorem, there exists a continuous map $\lambda : X \to [0, 1]$ such that

$$\begin{cases} 1 & \forall z \in X : d(z, x) \leq \frac{\delta}{4}; \\ 0, & \forall z \in X : d(z, x) \geq \frac{\delta}{2}. \end{cases}$$

Define $B : K \to K$ by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Since K is a closed convex subset of a complete $\mathbb{R}$-tree, $Bz \in K$ for all $z \in K$. In particular, B is continuous, by theorem (2.1). $B(K)$ is contained in a compact subset of X and $Bx = x$.

(1) For all $z \in K$ which $d(z, x) \geq \frac{\delta}{2}$ according to $Bz = Az$ we have
$$d(Az, Bz) = 0$$
(2) For all $z \in K$ which $d(z, x) \leq \frac{\delta}{4}$ according to $Bz = x$ we have
$$d(Az, Bz) = d(Az, x) < \varepsilon$$
(3) For all $z \in K$ which $d(z, x) \leq \frac{\delta}{2}$ we have
$$\begin{aligned} d(Bz, Az) &= d(\lambda(z)x \oplus (1 - \lambda(z))Az, Az) \\ &\leq \lambda(z)d(x, Az) + (1 - \lambda(z))d(Az, Az) \\ &\leq d(x, Az) \leq \varepsilon. \square \end{aligned}$$

Theorem 2.6. *Let (X, d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Let $B \in A$ and $\delta > 0$ be as guaranteed by theorem (2.5). Then for each $C \in A$ which $\rho(C, B) \leq \delta$, there exists $y \in K$ such that $Cy = y$ and $d(y, x) \leq \rho(B, C)$.*

Proof. By theorem (2.5), $\rho(A,B)\leq\varepsilon$ and $Bz=x$, for all $z\in K$ satisfies $d(z,x)\leq\delta$. Assume that $C\in A$ satisfies $\rho(C,B)\leq\delta$. Set

$$\Gamma:=\{z\in K: d(z,x)\leq\rho(C,B)\}.$$

Clearly, Γ is closed convex set, thus

$$\begin{aligned} d(x,Cz) &\leq d(x,Bz)+d(Bz,Cz)\\ &= d(Bz,Cz)\\ &\leq \rho(C,B), \end{aligned}$$

$Cz\in\Gamma$ for all $z\in\Gamma$. So $C(\Gamma)\subseteq\Gamma$, clearly $C(\Gamma)\subseteq C(X)$ is contained in a compact subset of X. Now by theorem (1.7) there exists $y\in K$ such that $Cy=y$.□

Theorem 2.7. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space. Then there exists a subset F of A which is a countable intersection of open subsets of (A,ρ) so that for each $A\in F$, A have the stable fixed point property.*

Proof. Let $A\in A$ and $\varepsilon>0$. By theorem (2.5) and (2.6) there exist $A_\varepsilon\in\Gamma$, $x_{A,\varepsilon}\in K$ and $\delta_{A,\varepsilon}\in(0,1)$ such that

$$\rho(A,A_\varepsilon)\leq\varepsilon,\quad A_\varepsilon z=x_{A,\varepsilon},$$

for all $z\in K$ satisfying $d(z,x_{A,\varepsilon})\leq\delta_{A,\varepsilon}$ and for all $C\in A$ satisfying $\rho(C,A_\varepsilon)\leq\delta_{A,\varepsilon}$ there exists $y\in K$ such that $Cy=y$ and $d(y,x_{A,\varepsilon})\leq\rho(C,A_\varepsilon)$. For each integer $i\geq 1$, set $U(A,\varepsilon,i):=\{C\in A:\rho(C,A_\varepsilon)<\frac{\delta_{A,\varepsilon}}{i}\}$. Define

$$F:=\bigcap_{i=1}^{\infty}\bigcup_{A\in A,\varepsilon\in(0,1)}U(A,\varepsilon,i).$$

Clearly F is a countable intersection of open subsets of (A,ρ).
Let $B\in F$, for all $i\geq 1$ there exists $A_i\in\Gamma$ and $\varepsilon_i\in(0,1)$ such that $B\in U(A_i,\varepsilon_i,i)$, then for all $i\geq 1$ there exists $y_i\in K$ such that

$$By_i=y_i,\quad d(y_i,x_{A_i,\varepsilon_i})\leq\rho(B,(A_i)_{\varepsilon_i})\leq\frac{\delta_{A,\varepsilon_i}}{i},$$

since $\{y_i\}_i\subseteq B(K)$ so there exists subsequence $\{y_{i_k}\}_k$ which $y_{i_k}\to x$ for some $x\in K$. Let $\varepsilon>0$ so there exists $k\in\mathbb{N}$ such that

$$\frac{1}{i_k}<\frac{\varepsilon}{8},\quad d(y_{i_k},x)\leq\frac{\varepsilon}{8},$$

then it follows from above

$$d(y_{i_k},x_{A_{i_k},\varepsilon_{i_k}})\leq\frac{1}{i_k}<\frac{\varepsilon}{8},$$

and

$$d(x,x_{A_{i_k},\varepsilon_{i_k}})\leq d(x,y_{i_k})+d(y_{i_k},x_{A_{i_k},\varepsilon_{i_k}})\leq\frac{\varepsilon}{4}.$$

Let $C \in U(A_{i_k}, \varepsilon_{i_k}, i_k)$ then there exists $z \in K$ such that $Cz = z$ and

$$d(z, x_{A_{i_k},\varepsilon_{i_k}}) \leq \rho(C, (A_{i_k})_{\varepsilon_{i_k}}) \leq \frac{1}{i_k} \leq \frac{\varepsilon}{8},$$

this implies that

$$d(z, x) \leq d(z, x_{A_{i_k},\varepsilon_{i_k}}) + d(x_{A_{i_k},\varepsilon_{i_k}}, x) \leq \frac{\varepsilon}{2}.\square$$

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