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SDP TECHNIQUE FOR THE TOTAL DOMINATING SET

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Abstract. In this paper, one of the most famous NP-complete problems in graph theory, the total dominating set problem, was investigated and a new quadratic integer programming model was presented. Finally, an SDP relaxation models are proposed. Finding the efficiency of the relaxation could be a future research direction.

1. Introduction

Consider an undirected and connected graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and E are respectively vertices and edges of G. The degree of vertex v_i is shown by $deg(v_i)$, and Δ stands for the maximum degree of the graph. A set $S \subseteq V$ is called dominating set of G if each vertex is a member of S or adjacent to a member of S. The set S is referred to as minimum dominating set if it has minimum cardinality among all dominating sets. The cardinality of minimum dominating set is called domination number and denoted by $\gamma(G)$. Domination number and its variations have been extensively studied in the literature. One of them is total domination number. A set S_t of vertices in a graph G is called a total dominating set if every vertex $v_i \in V$ is adjacent to an element of s_t. The size of total dominating set with minimum cardinality is denoted by $\gamma_t(G)$. For more details we refer the reader to [9].

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Dominating set and its variants are one of the classical problems in graph theory having important applications in many fields (e.g. [3, 4] for some recent applications). In [8], more than 1200 papers on different versions of dominating set problem are listed. Despite having a lot of application and theoretical attraction, Unfortunately, in [5] it has been shown the NP-completeness of dominaing set problem and subsequently the total dominating set problem. So, for any arbitary graph , it is not expected that the total dominating set will be found in reasonable time. To overcome to this challenge, there are several methods such linear relaxation, Greedy Algorithms and metaheuristics. In this paper, the semidefinite relaxation is applied to find an approximation solution for the total dominating set problem.
The semidefinite programming is a special case of convex optimization which linear objective function is optimized over the intersection of the cone of positive semidefinite matrices with linear constraints. Let $\mathbb{S}^n$ denote the set of symmetric $n \times n$ real matrices. The cone of symmetric positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$). $B-D \succeq 0$ ($B-D \succ 0$) means that $(B - D)$ is positive semidefinite (definite). Suppose that $A_1, \ldots, A_m$ are linearly independent matrices in $\mathbb{S}^n$; $C \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. The standard form of semidefinite programming problem is written as follows:

$$\begin{array}{ll} \min & \langle C, X\rangle \\ \text{s.t.} & \langle A_i, X\rangle \geq b_i \quad i = 1, 2, \ldots, n \\ & X \succeq 0 \end{array}$$

where $\langle B, D\rangle = \operatorname{tr}(B^t D) = \sum_{i,j} b_{ij} d_{ij}$. The semidefinite programming model can be solved in a polynomial time with an interior point method [1]. The interested reader is referred to [2] for a thorough discussion and applications of semidefinite programming. semidefinite programming relaxation is a powerful tool to approximate the optimal solution of some combinatorial problems. For example, dominating set [6] and maximum cut [7]. The good performance of semidefinite relaxation in these problems encouraged us to utilize this method to find an approximation of the k-tuple domination number.

2. Problem Description

The open neighborhood of a vertex v consists of the set of adjacent vertices to v, that is, $N(v) = \{w \in V | wv \in E\}$ and the closed neighborhood of is defined as $N[v] = N(v) \cup \{v\}$. The following labelling can be defined on V with respect to a subset $S \subseteq V$ as:

$$y(v_i) = \begin{cases} 1 & v \in S \\ -1 & v \notin V \end{cases}$$

For the sake of simplicity, we denote $y(v_i)$ by y_i and refer to a vertex with the label 1 as (+1)-vertex and as (-1)-vertex, otherwise. Further, $N(i)(N[i])$ stands for the open (closed) neighborhood of the vertex v_i. It is important to mention that a vertex in a total dominating set S_t is a (+1)-vertex induced by S_t. From the definition of labelling, it is clear that the objective function is $\frac{1}{2}\sum_{i=1}^{n}(1+y_i)$. The next lemma gives us valid inequlities for total dominating set.

Lemma 2.1. *$S_t \subseteq V$ is total dominating set if and only if it must satisfy in the following inequalities:*

$$\sum_{j\in N(i)}(1-y_iy_j)+\sum_{j\in N[i]}\frac{1+y_j}{2}\geq 2 \qquad i=1,2,\ldots,n. \tag{2.1}$$

Now, based on the (2.1), the quadratic integer programming model can be written as follows:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+y_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(1-y_iy_j)+\sum_{j\in N[i]}\frac{1+y_j}{2}\geq 2 & i=1,2,\ldots,n \\ & y_i\in\{-1,+1\} & i=1,2,\ldots,n \end{array} \tag{2.2}$$

Observe that the objective functions of (2.2) and part of inequalities are linear, while analyzing of our algorithms needs a quadratic objective function. To convert these linear functions to quadratic ones, a reference variable $y_0 \in \{-1,+1\}$ is introduced and problem (2.2) is rephrased as follows:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+y_0y_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(y_0^2-y_iy_j)+\sum_{j\in N[i]}\frac{y_0^2+y_0y_j}{2}\geq 2 & i=1,2,\ldots,n \\ & y_i\in\{-1,+1\} & i=0,1,2,\ldots,n \end{array} \tag{2.3}$$

Now suppose $\overline{y}=(y_0,y_1,\ldots y_n)$ be the optimal solution of (2.3). If $y_0=+1$ then $y=(y_1,\ldots y_n)$ is the optimal solution of (2.2) and if $y_0=-1$ then $y=(-y_1,\ldots-y_n)$ is the optimal solution of (2.2).

3. Semidefinite Relaxation

First, for $i=0,1,\ldots,n$, the variable y_i is substituted by an $(n+1)$-dimensional vector $u_i \in \mathbb{U}$ where $\mathbb{U}=\{(+1,0,\ldots,0),(-1,0,\ldots,0)\}$. Accordingly, the restriction $y_i\in\{-1,+1\}$ is replaced by $u_i \in \mathbb{U}$ and then problem (2.3) is adapted as:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^tu_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(u_0^tu_0-u_i^tu_j)+\sum_{j\in N[i]}\frac{u_0^tu_0+u_0^tu_j}{2}\geq 2 & i=1,2,\ldots,n \\ & u_i\mathbb{U} & i=0,1,2,\ldots,n \end{array} \tag{3.1}$$

Recall that $||u_i=1||$ for $u_i\in\mathbb{U}$ and this motivates to expand $\mathbb{U}$ to the standard $(n+1)$-dimensional unit sphere $\mathcal{S}^{n+1}=\{u\in\mathbb{R}^{n+1}|\ ||u||=1\}$, at the second step of the relaxation procedure. Thus, the following problem is obtained

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^tu_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(u_0^tu_0-u_i^tu_j)+\sum_{j\in N[i]}\frac{u_0^tu_0+u_0^tu_j}{2}\geq 2 & i=1,2,\ldots,n \\ & u_i^tu_i=1, \quad u_i\in\mathcal{S}^{n+1} & i=0,1,2,\ldots,n \end{array} \tag{3.2}$$

By introducing $X_{ij} = y_i y_j, E_{ij} = e_i e_j^t$ and $A_i = \sum_{j\in N(i)} \frac{1}{2}(2E_{00} - E_{ij} - Eji) + \sum_{j\in N[i]} \frac{1}{4}(2E_{00} - E_{0j} - Ej0)$, where e_i is the i-th standard unit vector of $\mathbb{R}^{n+1}$, the model (3.2) is converted to the following:

$$\begin{array}{lll} \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2 & i = 1, 2, \ldots, n \\ & X_{ii} = 1 & i = 0, 1, 2, \ldots, n \\ & \operatorname{rank}(X) = 1 & \\ & X \succeq 0 & \end{array} \tag{3.3}$$

where $C = (c_{ij}), c_{i0} = c_{0i} = \frac{1}{4}$ for $i = 1, \ldots, n$ and $c_{ij} = 0$ otherwise. By dropping the nonconvex constraint $\operatorname{rank}(X) = 1$ from (3.3), the semidefinite relaxation is formulated as:

$$\begin{array}{lll} \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2 & i = 1, 2, \ldots, n \\ & X_{ii} = 1 & i = 0, 1, 2, \ldots, n \\ & X \succeq 0 & \end{array} \tag{3.4}$$

The model (3.4) can be solved by interior point methods in CVX solver. Finally, the optimal solution of (3.4) just gives us a lower bound to total domination number.

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