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MAXIMUM WEIGHTED INDEPENDENT SET WITH UNCERTAIN WEIGHTS

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Abstract. The uncertainty theory from the viewpoint of Liu is a new way to deal with problems which some of parameters are not determinate. Especially, this theory is based on experts belifes and by introducing a measure in these belifes tries to overcome to uncertainty. Maximum weighted independent set problem is a classic combinatorial optimization problem and has wide range of application such as scheduling. It is proved that this is an NP-hrad problem and for arbitary graph, there are only approximate algorithms. In this paper, we investigate this problem with indeterministic weights and obtain an equivalant deministic integer programming model. Considering the concept of uncertainty distribution of an uncertain variable, one models is constructed based on α-chance method.

1. Introduction

When a real-world problem is modeled, the data are usualy considered indeterminate. In these situations, probability theory, fuzzy theory and theory of belief functions, also referred to as evidence theory or were introduced but unfortunatelly, these theories are not cover all problems. Recently, Baoding Liu proposed an axiomatic basis of uncertainty theory in 2007 [4] and refined it [5] in 2010. In this theory, the belifes of experts have essential role. In this

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paper, First, an integer programming model is presented for the maximum weighted independent set problem and a summary of uncertainty theory is explained and one model is discussed to solve this problem when the weights are uncertain.

2. An Integer Programming Model

$$\begin{array}{lll} \max & \sum_{i\in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \leq 1 & \forall (i,j) \in E, \\ & x_i \in \{0,1\}. & \end{array} \tag{2.1}$$

As already mentioned previously, the problem (2.1) is an NP-hard and no algorithm can find a solution in polynomial time unless P=NP. For obtaining an approximate solution of this model, semidefinite or linear relaxation is utilized. In the next section, a summary of the of uncertainty theory will be expressed from the viewpoint of Liu.

3. Uncertainty Theory

As already mentioned, uncertainty theory can be a potential tool for expressing experts' beliefs in mathematical language and using them. In this section, we point out some important concepts and features of this theory. For more details, refer the reader to [5].

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over it . Then $(\Gamma,\mathcal{L})$ is called measurable space and each member $\Lambda \in \mathcal{L}$ is called a measurable set or an *event*. Measurable space $(\Gamma,\mathcal{L})$ with uncertain measure $\mathcal{M}$ (this concept will be introduced later) is saied uncertainty space and is shown by $(\Gamma,\mathcal{L},\mathcal{M})$. A set function $\mathcal{M}$ over $\mathcal{L}$ is said to be an *uncertain measure* if it satisfies the following four axioms:

Axiom1 : (*Normality*) $\mathcal{M}\{\Gamma\}$=1 for the universal set Γ.
Axiom2 : (*Duality*) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}$=1 for any event Λ.
Axiom3 : (*Subadditivity*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom4 : (*Product*) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for integer $k \geq 1$. The product uncertain measure $\mathcal{M}$ is the one satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$ respectively and $\bigwedge$ stands for the minimum operator.

The function $f : (\Gamma, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$ is said to be measurable if for any Borel set B of real numbers, it holds $f^{-1}(B) = \{\gamma | f(\gamma) \in B\} \in \mathcal{L}$. An *uncertain variable* ξ is a measurable function on an uncertainty space. Also, ξ is considered nonegative if $\mathcal{M}\{\xi < 0\} = 0$ and positive if $\mathcal{M}\{\xi \leq 0\} = 0$. The next theorem talks about a fundamental and practical property in uncertainty theory.

Theorem 3.1. [5] *Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables. Further, let f be a real valued measurable function. Then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.*

For an uncertain variable ξ, *uncertainty distribution* Φ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. Different type of uncertain variables have been defined in the literature corresponding to different uncertainty distributions.

Definition 3.2. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be *independent* if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\},$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Theorem 3.3. [5] *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

In some cases, validity of an equality is not determined and α-chance model can be a useful interpretation for such situations. It is said that an equality $g(x, \xi) \leq 0$ holds with the belief degree α when $\mathcal{M}\{g(x, \xi) \leq 0\} \geq \alpha$. Determining the feasible region associated to such constraints in higher dimensional spaces is not straightforward. Next theorem presents an equivalent crisp constraint in specific circumstances.

Theorem 3.4. [5] *Let $g(x, \xi_1, \xi_2, \ldots, \xi_n)$ be a strictly increasing function with respect to $\xi_1, \ldots, \xi_k$, and strictly decreasing with respect to $\xi_{k+1}, \ldots, \xi_n$. Further, let $\xi_1, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \ldots, \Phi_n$, respectively. Then the relation $\mathcal{M}\{g(x, \xi_1, \xi_2, \ldots, \xi_n) \leq 0\} \geq \alpha$ holds if and only if*

$$g(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) \leq 0.$$

In this section, the model (2.1) is investigated when it's wheights are uncertain variables ξ_i.

$$\begin{array}{ll} \max & \sum_{i \in V} \xi_i x_i \\ \text{s.t.} & x_i + x_j \leq 1 \quad \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \implies \begin{array}{ll} \max & t \\ \text{s.t.} & t \leq \sum_{i \in V} \xi_i x_i \\ & x_i + x_j \leq 1 \qquad \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \tag{3.1}$$

$$\begin{array}{ll} \max & t \\ \text{s.t.} & \mathcal{M}\{t \le \sum_{i\in V} \xi_i x_i\} \ge \alpha \\ & x_i + x_j \le 1 \ \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \implies \begin{array}{ll} \max & t \\ \text{s.t.} & t \le \sum_{i\in V} \Phi_i^{-1}(\alpha) x_i \\ & x_i + x_j \le 1 \ \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \tag{3.2}$$

Finaly, by using theorem 3.3, the following deterministic model, is achieved.

$$\begin{array}{lll} \max & \sum_{i\in V} \Phi_i^{-1}(\alpha) x_i & \\ \text{s.t.} & x_i + x_j \le 1 & \forall (i,j) \in E, \\ & x_i \in \{0,1\}. & \end{array}$$

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