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SOME RESULTS IN REORDERED WOVEN FRAMES

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Abstract. In this paper we consider some results in woven frames. We study frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ for the Hilbert space H where π is a permutation function on $\mathcal{I}$, and give some results that $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is a frame for H, where σ is a subset of $\mathcal{I}$. In fact, we study the phase retrieval property and reordered weavings.

1. Introduction

Bemrose, Casazza, Grochenig, Lammers and Lynch in 2016 [**?**], introduced woven frames. Two frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ for the Hilbert space H are called woven if there exist universal bounds A, B with $0 < A \leq B < \infty$ such that for each $\sigma \subset \mathcal{I}$

$$A \parallel x \parallel^2 \leq \sum_{i\in\sigma} \mid \langle x, \phi_i\rangle \mid^2 + \sum_{i\in\sigma^c} \mid \langle x, \psi_i\rangle \mid^2 \leq B \parallel x \parallel^2,$$

for all $x \in H$. Maybe the basic motivation to create woven frames is a problem in distributed signal processing [**?**], particularly in wireless sensor networks where distributed signal processing occures by various frames. . For more details about frames, woven frames and some their applications we refer the redear to [**?**, **?**]. In 2019 P-woven frames are introduced [**?**], which are easier in applications than woven frames. The frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ are said to be P-woven if for some $\sigma \subset \mathcal{I}$ the family $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for H.

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* Speaker.

In this section at first, we mention the definition of frames and some basic properties of them in a separable Hilbert space H. After that we review the definition of woven frames. For more details, the reader is referred to [**?**, **?**]. Throughout the paper, H is a separable Hilbert space and $\mathcal{I}$ is a countable index set.

A family $\{\phi_i\}_{i\in\mathcal{I}}$ in H is a frame for H if there exist constants $0 < A \leq B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \leq \sum_{i\in\mathcal{I}} |\langle x, \phi_i\rangle|^2 \leq B\|x\|^2,$$

where A and B are called lower and upper frame bounds, respectively. If only B is assumed to exist, then $\{\phi_i\}_{i\in\mathcal{I}}$ is called a Bessel sequence. If $A = B$, then $\{\phi_i\}_{i\in\mathcal{I}}$ is called a tight frame, also it is called Parseval if $A = B = 1$. A frame $\{\phi_i\}_{i\in\mathcal{I}}$ is called exact when it ceases to be a frame when an arbitrary element is removed. Corresponding to each Bessel sequence $\{\phi_i\}_{i\in\mathcal{I}}$ in H, one can consider some important operators. The analysis operator $T : H \to l^2(\mathcal{I})$ is defined by $Tx = \{\langle x, \phi_i\rangle\}_{i\in\mathcal{I}}$. The synthesis operator $T^* : l^2(\mathcal{I}) \to H$ is given by $T^*\{c_i\}_{i\in\mathcal{I}} = \sum_{i\in\mathcal{I}} c_i\phi_i$, which is really the adjoin of the analysis operator T. In the case that $\{\phi_i\}_{i\in\mathcal{I}}$ is a frame for H, the frame operator $S : H \to H$ is defined by

$$Sx := T^*Tx = \sum_{i\in\mathcal{I}} \{\langle x, \phi_i\rangle\}\phi_i,$$

for each $x \in H$. It is well-known that, S is bounded, positive, self-adjoint and invertible. A frame which is a Schauder basis is called a Riesz basis. A frame which is not a Riesz basis is said to be overcomplete. A collection $\{\phi_i^j\}_{i\in I}$, $j = 1, \dots, n$, of frames for H is called a woven frame for H, if for each partition $\{\sigma_1, \dots, \sigma_n\}$ of $\mathcal{I}$ the family $\{g_i\}_{i\in\mathcal{I}}$ is a frame for H, where $g_i = \phi_i^j$ for all $i \in \sigma_j$, $j = 1, \dots, n$. Also, $\{\phi_i^j\}_{i\in I}$, $j = 1, \dots, n$, is called a P-woven frame for H, if there exists a partition $\{\sigma_1, \dots, \sigma_n\}$ of $\mathcal{I}$ such that the family $\{g_i\}_{i\in\mathcal{I}}$ is a framr for H.

2. Main results

A family $\{\phi_i\}_{i\in\mathcal{I}}$ in a Hilbert space H is said to be phase retrieval in H if whenever $x, y \in H$ satisfy

$$| < x, \phi_i > | = | < y, \phi_i > |,$$

for all $i \in \mathcal{I}$, then $x = \pm y$. A family $\{\phi_i\}_{i\in\mathcal{I}}$ for a separable infinite dimensional Hilbert space H is called has complement property, whenever for each $\mathcal{J} \subset \mathcal{I}$ one can deduce that $\overline{\text{span}}\{\phi_i\}_{i\in\mathcal{J}} = H$ or $\overline{\text{span}}\{\phi_i\}_{i\in\mathcal{I}\setminus\mathcal{J}} = H$. In finite dimensional case a set of vectors $\{\phi_i\}_{i=1}^{M}$ in an N-dimensional Hilbert space H is a full spark frame if either they are independent or if $M \geq N+1$, then they have spark $N + 1$. Also, a frame $\{\phi_i\}_{i\in\mathcal{I}}$ is called m-uniform excess for H, whenever for each $\sigma \subset \mathcal{I}$ with $|\sigma| = m$, $\{\phi_i\}_{i\in\mathcal{I}\setminus\sigma}$ is a frame for

H. The following proposition, is of much interesting when H is of infinite dimension.

Proposition 2.1. *Let H be an infinite dimensional separable Hilbert space. If $\{\phi_i\}_{i\in\mathcal{I}}$ does phase retrieval for H, then for each permutation function π on $\mathcal{I}$, the families $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven.*

Now, we state and prove the main theorem of this section.

Theorem 2.2. *Suppose that $\{\phi_i\}_{i=1}^M$ is a frame for an N-dimentional Hilbert space H. If $\{\phi_i\}_{i=1}^M$ is woven with any its reordering, then $\{\phi_i\}_{i=1}^M$ does phase retrieval in H.*

Corollary 2.3. *Suppose that $\{\phi_i\}_{i=1}^M$ is a frame for N-dimentional Hilbert space H where $M \geq 2N-1$. Then $\{\phi_i\}_{i=1}^M$ is woven with any its reordering if and only if $\{\phi_i\}_{i=1}^M$ does phase retrieval in H.*

Proposition 2.4. *Assume that $\{\phi_i\}_{i=1}^M$ is a frame for N-dimentional Hilbert space H and π is a permutation function on $\mathcal{I}$. Then for each $\sigma \subset \mathcal{I}$ the weavings $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ are full spark frames if and only if $\{\phi_i\}_{i=1}^M$ is full spark and $\pi = I_d$.*

The following example, shows that there are frames which are woven with each reordering of itself.

Example 2.5. Take the frame $\{\psi_i\}_{i=1}^M$ for the N-dimentional Hilbert space $H = \mathbb{C}^N$ as

$$\psi_i = \begin{cases} e_i & i = 1, 2, \ldots, N \\ \sum_{i=1}^N e_i & i = N+1, \ldots, M, \end{cases}$$

where $M \geq 2N$, and $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for H. Then for each permutation function π on $\mathcal{I}$, $\{\psi_i\}_{i=1}^M$ and $\{\psi_{\pi(i)}\}_{i=1}^M$ are woven.

3. Variable exponent version of the integral means

References

1. T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers and R. G. Lynch, Weaving frames, Operators and Matrices, 10 , 4 (2016), 1093-1116.
2. P.G. Casazza, and J. Kovačević, Uniform tight frames with erasures, preprint, 2001.
3. O. Christensen, An Introduction to Frames and Riesz Bases, 2nd edn, Birkhauser, Boston, (2015).
4. A. B. Hafshejani, M. A. Dehghan, P-woven frames, Journal of Mathematical Analysis and Applications, 479 (2019) 673-687.
5. B. Cyganek, Obeject detection and recognition in digital images (theory and practice), A John Wiley and Sons, 2013.