ANALYSIS OF A NONLINEAR OPERATOR IN THE FRAME OF EVOLUTIONARY GAME FOR DUAL RANDOM MARKETS

MAHDI KESHTKAR*, ELYAS SHIVANIAN

Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
keshtkarmahdi@gmail.com
Department of Applied Mathematics, Imam Khomeini International University, Qazvin 34148-96818, Iran
shivanian@sci.ikiu.ac.ir

ABSTRACT. We study the evolutionary dynamic of population, composed of two kinds of individuals, distributed randomly. In this game, evolution takes place in different periods of time between individuals frequently. We present a continuous nonlinear operator, which describes and fulfills this discrete time evolutionary game. Furthermore, we set up the adequate mathematical framework to obtain the fixed point of this operator. Based on this equilibrium state of the evolutionary operator, it is proved the possessions of the individuals vanish.

1. INTRODUCTION

The population is composed of individuals which can have two differing "life strategies", and the success or failure of these strategies has a direct consequence upon the continued reproductive success of the individuals[1]. In the last years, some different techniques and models, from statistical

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physics, have been applied successfully to some real data observed in economy [2]. Different models have been used to explain the origin of these wealth distributions. A kind of models considering this unknowledge associated to markets are the gas-like models [3]. In order to explain the two different types of statistical behavior before mentioned, different gas-like models have been proposed. On the one hand, the exponential distribution can be obtained by supposing a gas of agents that trade with money in binary collisions, or in first-neighbor interaction, and where the agents are selected in a random, deterministic or chaotic way [4].
Alongside this approach, however, there has also emerged a significant literature that seeks to extend such evolutionary dynamics to games with nonlinear operators [5]. This paper seeks to address further this lacuna and provide foundations to the nonlinear operators for games with two kinds of player (or equivalently agent) that compete together for obtaining wealth. We construct an operator that governs the discrete time evolution of the wealth distribution in population that is composed of two kinds of individuals, say type F (Fight) and type Y (Yield), distributed randomly which interact by pairs and exchange their money in a random way. It is shown the previous model [6] describing exponential wealth distribution in a random market is the special case of present model if we omit the individuals of type F (Fight).
The rest of the paper is structured as follows: In section 2, we introduce the nonlinear model for population games with two kind of players, section 3 establishes the fundamental mathematical properties of the operator T. Finally, a conclusion is given in section 4.

2. Nonlinear model

We consider an ensemble of economic agents (individuals or equivalently players) in two categories (Yields and Fights) which trading their money by pairs in a random manner. There are three kinds of trading between players (individuals) which are as follows by first, second and third cases. Before that, in all following cases, we notice that ε is a number in the interval $(0,1)$. Moreover, the agents $(i,\ j)$ are randomly chosen. Finally, their initial money (m_i, m_j), at time$t$, is transformed after the interaction into $\left(m_i', m_j'\right)$at time $t+1$.
The continuous version of this model considers the evolution of an initial wealth distribution $p_0(m)$ at each time step n under the action of an operator T. Thus, the system evolves from time n to time $n+1$ to asymptotically reach the equilibrium state of zero wealth, i.e.

$$\lim_{n\to\infty} T^n(p_0(m)) \to 0 \tag{2.1}$$

The derivation of the operator $T$is as follows [6]. Suppose that $p_n$is the wealth distribution in the ensemble at time n. The probability to have a quantity of money x at time $n+1$ will be the sum of the probabilities of all those pairs of agents (u, v) able to produce the quantity x after their interaction, that is, all the pairs verifying $u+v > x$ in the first case and third case and all the pairs verifying $u + v - c > x$ in the second case. Thus, the probability that two of these agents with money (u, v) interact between them is $p_n(u)*p_n(v)$. Their exchange is totally random and then they can give rise with equal probability to any value $x$comprised in the interval $(0, u + v)$ for the first and third cases, and $(0, u + v)$ for the second case. Therefore, the probability to obtain a particular x (with $x < u+v$ *or* $x+c < u+v$) for the interacting pair will be $\frac{p_n(u)*p_n(v)}{u+v}$. Then, we present the general continuous nonlinear operator for discrete time evolutionary game (it is more general than that considered in [8-10]), which comes from the combination of the above three cases, as following form

$$\begin{aligned} p_{n+1}(x) &= Tp_n(x) \\ &= \frac{1}{3}\iint_{u+v>x} \frac{p_n(u)p_n(v)}{u+v}dudv + \frac{1}{3}\iint_{u+v>x} \frac{p_n(u)p_n(v)}{u+v}dudv \\ &+\frac{1}{3}\iint_{u+v>x+c} \frac{p_n(u)p_n(v)}{u+v}dudv \end{aligned} \tag{2.2}$$

The right hand terms are related to cases (Yield, Yield), (Fight, Yield) and (Fight, Fight), respectively. if we assume $c = 0$ where we look to Yield and Fight as the same player.

3. Mathematical properties of the operator

Definition 3.1. We introduce the space L_1^+ of positive functions (wealth distributions) in the interval $[0, \infty)$,

$$L_1^+[0, \infty) = \{y : [0, \infty) \to R^+ \cup \{0\}, \|y\| < \infty\}, \tag{3.1}$$

with norm-1

$$\|y\| = \int_0^\infty y(x)dx. \tag{3.2}$$

In particular, consider the subset of $L_1^+[0, \infty)$ i.e. the unit sphere

$$B = \left\{y \in L_1^+[0, \infty), \quad \|y\| = 1\right\}.$$

Definition 3.2. For $x \geq 0$and $y \in L_1^+[0, \infty)$ the action of operator Ton y is defined by

$$\begin{aligned} T(y(x)) &= \frac{1}{3}\iint_{S(x)} \frac{y(u)y(v)}{u+v}dudv + \frac{1}{3}\iint_{S(x)} \frac{y(u)y(v)}{u+v}dudv \\ &+ \frac{1}{3}\iint_{S_c(x)} \frac{y(u)y(v)}{u+v}dudv \end{aligned} \tag{3.3}$$

where $S(x)$and $S_c(x)$ are the regions of the plane representing the pairs of agents (u, v) which can generate a richness x after their trading, i.e.

$$S(x) = \{(u, v), u, v > 0, u + v > x\}$$

$$S_c(x) = \{(u, v), u, v > 0, u + v > x + c\}$$

If there was not cost c for any of agents which are trading money, operator Tdefined in (3.3) conserves the norm $(|| \cdot ||)$, i.e. T maintains the total number of agents (those agents which are active in game) of the system, i.e.

$\|Tp\| = \|p\| = 1$, that by extension implies the conservation of the total richness of the system. However, in the present model, T does not maintain the total numbers of agents, which are active in game, because some of them lose their total money, in the other word, their money vanish. Therefore, it is credible to expect $\|Tp\| < \|p\|$.

Lemma 3.3. *We claim that for anyy* $\in L_1^+[0,\infty)$ *and* $c > 0$,

$$c\int_c^\infty \int_c^\infty \frac{y(u)y(v)}{u+v}dudv < \|y\|^2 .$$

Theorem 4. *For any* $y \in L_1^+[0,\infty)$ *we have*

$$||T\mathrm{y}|| \le ||y||^2 - c\int_c^\infty \int_c^\infty \frac{y(u)y(v)}{u+v}dudv.$$

It means that the number of active agents in the economic system is not conserved in time .i.e. in the unit sphere B, we observe that if $y \in B$ then $Ty \notin B$.

Theorem 3.4. *Consider the unit sphere*$B = \{y \in L_1^+[0,\infty),\ \|y\| = 1\}$, *if* $y \in B$ *then* $y_{n+1}(x) = Ty_n(x) = T^n y(x)$ *is a decreasing operator respected to norm-1 while it remains always* $y_n(x) \in L_1^+[0,\infty)$.

Corollary 3.5. *Suppose that* $y \in B$ *then the system asymptotically reach the equilibrium distribution 0, i.e.*

$$\lim_{n\to\infty} T^n(y(x)) \to 0.$$

4. Conclusion

Summarizing, in population that is composed of two kinds of individual which compete together in dual market randomly by trading their money. We have introduced a continuous nonlinear operator and then obtained the fixed point of this operator. Based on this equilibrium state of the evolutionary operator, it has been proved the possession of the individuals vanishes.

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