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TOPOLOGICALLY TRANSITIVITY ON NON-SEPARABLE BANACH SPACES

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ABSTRACT. When X is an infinite-dimensional Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X, then we will consider topological transitive operators in $B(X)$. In this paper, we will investigate the density of the range of a topological transitive operator on X.

1. INTRODUCTION AND PRELIMINARIES

Assume that X is a Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X. Consider an operator $T \in B(X)$. If for every pair U, V of nonempty open subsets of X, there is a positive integer n so that subset $T^n(U) \cap V$ is nonempty, then the operator T is topological transitive. The first example of topological transitive operators was presented by Birkhoff in [4]. If the underlying space is considered as a separable Banach space, then it is a simple matter to see that topological transitivity is equivalent to hypercyclicity. To be more clear, if B is a subset of X, then the orbit of B under T is the set $orb(T, B) = \{T^n x;\ x \in B, n = 0, 1, 2, \cdots\}$. if B is a singleton $\{x\}$ and $\overline{orb(T, B)} = X$, then T is called a hypercyclic operator and x is a hypercyclic vector for T. If $B = \{\lambda x;\ \lambda \in \mathbb{C}\}$ for some

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vector $x \in X$ and $orb(T, B)$ is a dense subset of X, then this operator is said to be a supercyclic operator and the vector x is a supercyclic vector for T. Note that, since $\{T^n x;\ x \in X, n = 0, 1, 2, \cdots\} \subseteq T(X)$, so a topological transitive operator T on a separable Banach space has dense range. We recall that, there is no hypercyclic operator on a finite-dimensional Banach space. On the other side, Ansari [1] showed that every infinite-dimensional separable Banach space admits a topological transitive operator. Hence, in the following X is an infinite-dimensional Banach space.

It is worth pointing out that when X is an infinite-dimensional non-separable Banach space, then obviously X cannot support hypercyclic operators. However, it is well known that topologically transitive operators may exist in X, see for instance [5].

Now it is natural to raise the following question.

Problem. Let T be a topological transitive operator T on a non-separable Banach space X. Dose it have dense range?.

In the next section we will give positive answer to this question as the main result of this paper. For details and references on topological transitive operators on non-separable and separable Banach spaces see [2] and [3].

2. Topological transitivity and whose equivalent assertions

The second section deals with some assertions which are equivalent to topological transitivity. We emphasize that in this section the underlying space X is an arbitrary Banach space, so it may be a non-separable Banach space. Then we will give two different proofs of the density of the range of a topological transitive operator on X.

Theorem 2.1. *Let T be an operator on a Banach space X. Then the following are equivalent.*
i) T is topological transitive,
ii) $\overline{\bigcup_{n=0}^{\infty} T^n(U)} = X$, whenever U is an arbitrary open subset of X,
iii) $\overline{\bigcup_{n=0}^{\infty} T^{-n}(U)} = X$, whenever U is an arbitrary open subset of X,
iv) every proper open T^{-1}-invariant subset of X is dense in X,
v) every proper closed T-invariant subset of X is nowhere dense in X.

Proof. Since $T^n(U) \cap V \neq \emptyset$ and $T^{-n}(V) \cap U \neq \emptyset$ are equivalent, so it is evident that the assertions *(i)*, *(ii)* and *(iii)* are equivalent. Thus we only need to prove *(i)* $\Longleftrightarrow$ *(iv)*, because *(i)* $\Longleftrightarrow$ *(v)* can be proved in much the same way as *(i)* $\Longleftrightarrow$ *(iv)*. For this goal, in the first step assume that U is an open subset of X and $T^{-1}(U) \subseteq U$. To obtain a contradiction, suppose that there exists an $x \in X \setminus \overline{U}$. Now if we cosider a neighbourhood V_x of x, then *(i)* implies that $T^n(V_x) \cap U \neq \emptyset$, for some $n \in \mathbb{N}$. Consequently, the contradiction

$$\emptyset \neq T^{-n}(U) \cap V_x \subseteq T^{-n+1}(U) \cap V_x \subseteq \cdots U \cap V_x = \emptyset,$$

shows that *(i)* implies *(iv)*.
Conversely, again to obtain a contradiction suppose that

$$\left(\bigcup_{n=0}^{\infty} T^{-n}(U)\right) \cap V = \emptyset,$$

for some non-empty open subsets U, V of X. This means that the open subset $\widehat{U} := \bigcup_{n=0}^{\infty} T^{-n}(U)$ is not dense in X. The assertion *(iv)* implies that $T^{-1}(\widehat{U}) \nsubseteq \widehat{U}$ which is a contradiction because it is easy to check that $T^{-1}(\widehat{U}) \subseteq \widehat{U}$. Therefore the proof of *(iv)* $\Rightarrow$ *(i)* is completed.

□

We can get the following corollary from the assertion *(iii)* in the previous theorem.

Corollary 2.2. *Every topological transitive operator on a separable or non-separable has dense range.*

Proof. Let V be an arbitrary open subset of X. the assertion *(iii)* implies that $T^{-m}(V) \cap V \neq \emptyset$, for some $m \in \mathbb{N}$, so there exists a vector $x \in V$ such that $T^m x \in V$. This means that $T^m x \in T(X)$ and consequently $V \cap T(X)$ is non-empty. Therefore $T(X)$ is dense in X. □

It is interesting to know that the above result can be derived from *(v)*. Thus we give a different proof of the above corollary.

Proof. Assume that T is topological transitive operator and also assume that λ is an eigenvalue of T^*. If x^* is a corresponding eigenvector to λ, then one of the subsets $\{x : |x^*(x)| \geq 1\}$ or $\{x : |x^*(x)| \leq 1\}$ is an invariant under T with non-empty interior. This contrary to the assertion *(v)* and consequently $\sigma_p(T^*) = \emptyset$. Since $\sigma_p(T^*) = \emptyset$ is equivalent to the density of the range of T, so the proof is completed. □

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