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HERMITE-HADAMARD INTEGRAL INEQUALITY FOR SOME TYPES OF CONVEX FUNCTIONS

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Abstract. In this paper, we verify Hermite-Hadamard integral inequality on some types of convex functions. Previous results are some part of our consequences.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $x, y \in I$. Then (trapezium inequality)

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This double inequality is known in the literature as the Hermite-Hadamard (HH) integral inequality for convex functions.

2. Preliminary

Definition 2.1 ([12]). Let $m, t, \alpha \in [0, 1]$. Then the real number set $C \subseteq \mathbb{R}$ is said to be

(1) convex if $tx + (1-t)y \in C$;
(2) m-convex if $tx + (1-t)my \in C$;
(3) (α, m)-convex if $t^\alpha x + (1-t^\alpha)my \in C$;

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for all $x, y \in C$ and $t, m \in [0,1]$.

Definition 2.2 ([3, 4, 9, 10, 12]). Let $m \in [0,1]$ and $C \subseteq \mathbb{R}$. A function $f : C \subset \mathbb{R} \to \mathbb{R}$ is said to be an

(1) convex, if C be a convex set and

$$f(tx + (1-t)y] \leq tf(x) + (1-t)f(y);$$

(2) m-convex, if C be a m-convex set and

$$f(tx + (1-t)my] \leq tf(x) + (1-t)mf(y);$$

(3) (α, m)-convex, if C be a (α, m)-convex set and

$$f(tx + (1-t)my] \leq t^{\alpha} f(x) + (1-t^{\alpha})mf(y);$$

(4) f is concave if $-f$ is convex;
(5) f is m-concave if $-f$ is m-convex.
(6) $f : [a,b] \to \mathbb{R}$ is star shaped if $f(tx) \leq tf(x)$ for all $t \in [0,1]$ and $x \in [a,b]$.

for all $x, y \in C$ and $t, m \in [0,1]$.

Remark 2.3. ([10, 7])

(1) When $t = 1$, we get $f(my) \leq mf(y)$ for all $x, y \in I$, means the function f is sub-homogeneous.
(2) If f was convex function and $m = 1$, it would be m-convex function.

Lemma 2.4. ([4, 7])

(1) *If $f : C \to \mathbb{R}$ is m-convex and $0 \leq n < m \leq 1$, then f is n-convex.*
(2) *Let $f, g : [a,b] \to \mathbb{R}$, $a \geq 0$. If f is n-convex and g is m-convex, with $n \leq m$, then $f + g$ and αf, $\alpha \geq 0$ a constant, are n-convex.*
(3) *Let $f : [0,a] \to \mathbb{R}$, $g : [0,b] \to \mathbb{R}$, with* $\text{renge}(f) \subseteq [0,b]$. *If f and g are m-convex and g is increasing, then $g \circ f$ is m-convex on $[0,a]$.*
(4) *If $f, g : [0,a] \to \mathbb{R}$ are both nonnegative, increasing and m-convex, then fg is m-convex.*

3. Main results

Put $\text{co}(A) = \{f : f \text{ is convex}\}$ and $\text{co}_m(B) = \{f : f \text{ is } m-\text{convex}\}$. So $\text{co}_m(B) \subsetneqq \text{co}(A)$, See more detail in [1].

Theorem 3.1. *Let $m \in [0,1]$ and $C \subseteq \mathbb{R}$ and function $f : C \subset \mathbb{R} \to \mathbb{R}$ be a m-convex function on an interval C and $a, b \in C$. If $a + mb = r + s$ for every r and s. Then*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{1}{2}\left(\frac{1}{r-a}\int_a^r f(u)du + m\frac{1}{mb-s}\int_s^{mb} f(u)du\right)$$
$$\leq \frac{f(r) + mf(s)}{2} + \frac{f(a) + mf(mb)}{2}.$$

Corollary 3.2. *By hypothesis of Threorem 3.1 we have:*

$$\frac{1}{2}\left(\frac{1}{r-a}\int_a^r f(u)du + m\frac{1}{mb-s}\int_s^{mb} f(u)du\right) \leq \frac{f(r)+f(a)}{2} + m\frac{f(s)+mf(b)}{2}.$$

Corollary 3.3.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{4} + \frac{1}{2}f\left(\frac{a+b}{2}\right).$$

An other version of Theorem 3.1:

Theorem 3.4. *Let $m \in [0,1]$ and $C \subseteq \mathbb{R}$ and function $f : C \subset \mathbb{R} \to \mathbb{R}$ be a m-convex function on an interval C and $a, b \in C$. Then*

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{mb-a}\left(\int_a^{\frac{mb+a}{2}} f(u)du + m\int_{\frac{mb+a}{2}}^{mb} f(u)du\right) \\ &\leq \frac{f(a)+mf(b)}{2}\left(\frac{m+1}{4}\right) + \frac{f(a)+m^2f(b)}{4} \\ &= \frac{(3m+1)(mf(b)) + (m+3)f(a)}{8}. \end{aligned}$$

Corollary 3.5. *By hypothesis of Theorem 3.4:*

$$\frac{1}{mb-a}\left(\int_a^{\frac{mb+a}{2}} f(u)du + m\int_{\frac{mb+a}{2}}^{mb} f(u)du\right) \leq \frac{(3m+1)(mf(b)) + (m+3)f(a)}{8}.$$

If we put $m = 1$ in Theorem 3.4, then we will find Hermite-–Hadamard (HH) integral inequality.

For more details and some of related references see [2, 5, 6, 8, 11, 13].

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