The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

HYPERBOLICITY AND SHADOWING PROPERTY IN ORLICZ SPACES

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Abstract. In the present paper we provide some equivalent conditions for composition operators to have shadowing property on Orlicz space $L^{\Phi}(\mu)$. Also, we obtain that for the composition operators on Orlicz spaces the notions of generalized hyperbolicity and the shadowing property coincide. The results of this paper extends similar results on L^p-spaces.

1. Introduction

In last two decades mathematicians have obtained plenty of interesting results in linear dynamics concerning dynamical properties such as transitivity, mixing, Li-Yorke and many others. Finding relations between expansivity, hyperbolicity,the shadoing property and structural stability is one of important questions in linear dynamics. In [2] the authors have shown that for the class of bounded composition operators on $L^p(\mu)$, the notion of generalized hyperbolicity and the shadowing property coincide. Indeed they provide some sufficient and necessary conditions for composition operators to have the shadowing property. In this paper we extend these results for composition operators on Orlicz spaces. Indeed we provide some equivalent conditions for composition operators to have shadowing property on Orlicz space $L^{\Phi}(\mu)$. Also, we obtain that for the composition operators on Orlicz

2020 *Mathematics Subject Classification.* Primary 37D05; Secondary 47A16.

Key words and phrases. Shadowing property, Hyperbolicity, Dissipative systems, Orlicz spaces, Composition operators.

spaces the notions of generalized hyperbolicity and the shadowing property coincide.

Here for the convenience of the reader, we recall some essential facts on Orlicz spaces for later use. For more details on Orlicz spaces, see [3, 4].

A function $\Phi : \mathbb{R} \to [0, \infty]$ is called a *Young's function* if Φ is convex, $\Phi(-x) = \Phi(x)$, $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = +\infty$. With each *Young's function* Φ, one can associate another convex function $\Psi : \mathbb{R} \to [0, \infty]$ having similar properties defined by

$$\Psi(y) = \sup\{x \mid y \mid -\Phi(x) : x \geq 0\}, \qquad y \in \mathbb{R}.$$

The function Ψ is called the *complementary function* to Φ. For a *Young's function* Φ, let $\rho_\Phi : L^\Phi(\mu) \to \mathbb{R}^+$ such that $\rho_\Phi(f) = \int_X \Phi(f)d\mu$ for all $f \in L^\Phi(\mu)$. Then the space

$$L^\Phi(\mu) = \left\{f \in L^0(\mathcal{F}) : \exists k > 0, \rho_\Phi(kf) < \infty\right\}$$

is called an Orlicz space. Define the functional

$$N_\Phi(f) = \inf\{k > 0 : \rho_\Phi(\frac{f}{k}) \leq 1\}.$$

The functional $N_\Phi(.)$ is a norm on $L^\Phi(\mu)$ and is called *guage norm*(or Luxemburge norm). Also, $(L^\Phi(\mu), N_\Phi(.))$ is a normed linear space. If almost every where equal functions are identified, then $(L^\Phi(\mu), N_\Phi(.))$ is a Banach space, the basic measure space $(X, \mathcal{F}, \mu)$ is unrestricted. Hence every element of $L^\Phi(\mu)$ is a class of measurable functions that are almost every where equal.

Throughout this paper, $(X, \mathcal{F}, \mu)$ will be a measure space, that is, X is a set, $\mathcal{F}$ is a sigma algebra on X and μ is a positive measure on $\mathcal{F}$. Also, we assume that $\varphi : X \to X$ is a non-singular measurable transformation, that is, $\varphi^{-1}(F) \in \mathcal{F}$, for every $F \in \mathcal{F}$ and $\mu(\varphi^{-1}(F)) = 0$, if $\mu(F) = 0$. Moreover, for the non-singular measurable transformation $\varphi : X \to X$, if there exists a positive constant c for which

$$\mu(\varphi^{-1}(F)) \leq c\mu(F), \quad \text{for every} F \in \mathcal{F}, \tag{1.1}$$

then the linear operator

$$C_\varphi : L^\Phi(\mu) \to L^\Phi(\mu), \qquad C_\varphi(f) = f \circ \varphi,$$

is well-defined and continuous on the Orlicz space $L^\Phi(\mu)$ and is called composition operator. For more details on composition operators on Orlicz spaces one can refer to [1].

2. Hyperbolicity and shadowing property

In this section first we rewrite the definitions of wandering set, dissipative system and dissipative system of bounded distortion in Orlicz spaces, that were defined for L_p spaces in [2].

Definition 2.1. Let $(X,\mathcal{F},\mu)$ be a measure space and $\varphi : X \to X$ be an invertible non-singular transformation. A measurable set $W \subseteq X$ is called a wandering set for φ if the sets $\{\varphi^{-n}(W)\}_{n\in\mathbb{Z}}$ are pair-wise disjoint.

Definition 2.2. Let $(X,\mathcal{F},\mu)$ be a measure space and $\varphi : X \to X$ be an invertible non-singular transformation. The quadruple $(X,\mathcal{F},\mu,\varphi)$ is called

- a dissipative system generated by W, if $X = \dot{\cup}_{k\in\mathbb{Z}}\varphi^k(W)$ for some $W \in \mathcal{F}$ with $0 < \mu(W) < \infty$ (the symbol $\dot{\cup}$ denotes pairwise disjoint union);
- a dissipative system, of bounded distortion, generated by W, if there exists $K > 0$, such that

$$\frac{1}{K}N_\Phi(C^k_\varphi(\chi_W))N_\Phi(\chi_F) \leq N_\Phi(C^k_\varphi(\chi_F))N_\Phi(\chi_W) \leq KN_\Phi(C^k_\varphi(\chi_W))N_\Phi(\chi_F), \tag{2.1}$$

for all $k \in \mathbb{Z}$ and $F \in \mathcal{F}_W = \{F \cap W : F \in \mathcal{F}\}$. If we replace N_Φ by the norm of L_p, then we will have the bounded distortion property in L_p-spaces.(Definition 2.6.4, [2])

Definition 2.3. A composition dynamical system $(X,\mathcal{F},\mu,\varphi,C_\varphi)$ is called

- dissipative composition dynamical system, generated by W, if $(X,\mathcal{F},\mu,\varphi)$ is a dissipative system generated by W;
- dissipative composition dynamical system, of bounded distortion, generated by W, if $(X,\mathcal{F},\mu,\varphi)$ is a dissipative system of bounded distortion, generated by W.

In the following we have a proposition in Orlicz spaces similar to the Proposition 2.6.5 of [2].

Proposition 2.4. *Let $(X,\mathcal{F},\mu,\varphi)$ be a dissipative system of bounded distortion, generated by W. Then the followings hold:*

(1) *There exists $H > 0$ such that*

$$\frac{1}{H}\frac{N_\Phi(C^{t+s}_\varphi(\chi_W))}{N_\Phi(C^s_\varphi(\chi_W))} \leq \frac{N_\Phi(C^{t+s}_\varphi(\chi_F))}{N_\Phi(C^s_\varphi(\chi_F))} \leq H\frac{N_\Phi(C^{t+s}_\varphi(\chi_W))}{N_\Phi(C^s_\varphi(\chi_W))}, \tag{2.2}$$

for all $F \in \mathcal{F}_W$ with $\mu(F) > 0$ and all $s,t \in \mathbb{Z}$.

(2) *Let $\Phi \in \Delta_2$. If*

$$\sup\{\frac{N_\Phi(C^{k-1}_\varphi(\chi_W))}{N_\Phi(C^k_\varphi(\chi_W))}, \frac{N_\Phi(C^{k+1}_\varphi(\chi_W))}{N_\Phi(C^k_\varphi(\chi_W))} : k \in \mathbb{Z}\}$$

is finite and φ is bijective, then φ and φ^{-1} satisfy 1.1 condition.

Let $h_k = \frac{d(\mu\circ\varphi^{-k})}{d\mu}$ denotes the Radon-Nikodym derivative of $\mu\circ\varphi^{-k}$ with respect to μ, where $\mu\circ\varphi^{-k}(F) = \mu(\varphi^{-k}(F))$, for every $F \in \mathcal{F}$.

Proposition 2.5. *Let $(X,\mathcal{F},\mu,\varphi)$ be a dissipative system generated by W and φ is injective. We put $m_k = ess\,inf_{x\in W}h_k(x)$, and $M_k = ess\,sup_{x\in W}h_k(x)$. Without lose of generality we can assume that $m_k \leq 1$ and $M_k \geq 1$. If the sequence $\{\frac{M_k}{m_k}\}_{k\in\mathbb{Z}}$ is bounded, then φ is of bounded distortion on W.*

Definition 2.6. Let φ, φ^{-1} satisfy the condition 1.1 and $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system generated by W. We define the following conditions

$$\overline{\lim}_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})} \right)^{\frac{1}{n}} < 1 \tag{2.3}$$

$$\underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{Z}} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})} \right)^{\frac{1}{n}} > 1 \tag{2.4}$$

$$\overline{\lim}_{n\to\infty} \sup_{k\in-\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k-n}(W))})} \right)^{\frac{1}{n}} < 1 \text{ and } \underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})} \right)^{\frac{1}{n}} > 1 \tag{2.5}$$

From now on we assume that $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system generated by W such that the associated composition operator C_φ is an invertible operator on Orlicz space $L^\Phi(\mu)$.

Theorem 2.7. *If $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system of bounded distortion generated by W, then the following hold.*

1. *If the condition 2.3 is satisfied, then C_φ is a proper contraction under an equivalent norm, i.e., $r(C_\varphi) < 1$.*
2. *If the condition 2.4 holds, then C_φ is a proper dilation under an equivalent norm, i.e., $r(C_\varphi^{-1}) < 1$.*
3. *If the condition 2.5 is satisfied, then C_φ is a generalized hyperbolic operator.*

So C_φ has the shadowing property in all three cases.

References

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