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CONFORMABLE STURM–LIOUVILLE PROBLEM WITH TRANSMISSION CONDITIONS

MOHAMMAD SHAHRIARI

Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran.
shahriari@maragheh.ac.ir

Abstract. In this manuscript, we study the Sturm–Liouville problem with conformable fractional differential operators of order α, $0.5 < \alpha \le 1$ and finite number of interior discontinuous conditions. The asymptotic formulas of solutions, eigenvalues and eigenfunctions of the problem are calculated.

1. Introduction

Sturm–Liouville equation is one of the most important problems in mathematics, physics and engineering. This problem arises in modeling of many systems in vibration theory, quantum mechanics, hydrodynamic and so on [1, 2]. The classical Sturm–Liouville equation is a second order ordinary differential equation of the following form:

$$y'' + (\lambda - q(x))y = 0, \quad 0 < x < \pi, \tag{1.1}$$

where $q(x)$ is the potential function and λ is a parameter. For equation (1.1) two boundary conditions at end points are considered. Equation (1.1) with boundary conditions are called Sturm–Liouville problems (SLP). Fractional Sturm-Liouville problems are different from those usually defined in this literature, i.e. the ordinary derivatives in a traditional Sturm-Liouville problem are replaced with fractional derivatives or derivatives of fractional

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order. These types of FSLP play a significant role in various areas of science, engineering, and mathematics [3, 4, 5]. In this note, we study the asymptotic form of characteristic function, eigenvalues, and eigenfunctions of conformable fractional Sturm–Liouville problem (CFSLP).

2. Asymptotic form of solutions and eigenvalues

In this section, we give definition and some theorems of the conformable fractional(CF) derivative such that one can found in [6, 7]. In what follows, we always take $D^\alpha_x = D^\alpha$.

Definition 2.1. For the function $f:[0,\infty)\to\mathbb{R}$, the CF derivative of f of order $\alpha\in(0,1]$ defined by:

$$D^\alpha f(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon},$$

for all $x>0$, and

$$D^\alpha f(0)=\lim_{x\to 0^+}D^\alpha f(x).$$

If f is a differentiable function, then

$$D^\alpha f(x)=x^{1-\alpha}f'(x).$$

If $D^\alpha f(x_0)$ exists and is finite. Then the function f is α-differentiable at x_0.

Definition 2.2. The conformable integral of function f of order α is defined as:

$$J^\alpha f(x)=\int_0^x f(s)\mathrm{d}_\alpha s=\int_0^x s^{\alpha-1}f(s)\mathrm{d}s,\quad x>0.$$

where, the integrals are in Riemann setting.

Definition 2.3. For a real number $1\le p<\infty$ and $\alpha>0$, the space $L^\alpha_p(0,a)$ is defined by

$$L^\alpha_p(0,a)=\left\{f:[0,a]\to\mathbb{R},\ \left(\int_0^a |g(t)|^p d_\alpha t\right)^{1/p}<\infty\right\}.$$

Let us consider the CFSLP

$$\ell_\alpha y := -D^\alpha D^\alpha y+qy=\lambda y \tag{2.1}$$

with boundary conditions

$$\begin{aligned} B_1(y) &:= D^\alpha y(0)+h\,y(0)=0,\\ B_2(y) &:= D^\alpha y(\pi)+H\,y(\pi)=0, \end{aligned} \tag{2.2}$$

and finite number of transmission conditions

$$\begin{aligned} U_i(y) &:= y(d_i+)-a_iy(d_i-)=0,\\ V_i(y) &:= D^\alpha y(d_i+)-b_iD^\alpha y(d_i-)-c_iy(d_i-)=0, \end{aligned} \tag{2.3}$$

for $i = 1, 2, \ldots, m-1$ and $\frac{1}{2} < \alpha \leq 1$. The parameters h, H and a_i, b_i, c_i, $d_i \in (0, \pi)$ are real numbers. We denote the problem (2.1)–(2.3) with $L_\alpha = L_\alpha(q(x); h; H; d_i)$. Consider the weighted inner product

$$\langle f, g\rangle_T := \int_0^\pi f(t)\overline{g(t)}w(t)\mathrm{d}_\alpha t,$$

where $f, g \in L_2^\alpha((0,\pi); w)$ and $w(t)$ is the weight function of the form

$$w(t) = \begin{cases} 1, & 0 \leq t < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < t < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < t \leq \pi. \end{cases}$$

Note that $T := L_\alpha^2((0,\pi); w)$ is a Hilbert space with the norm $\|f\|_T = \langle f, f\rangle_T^{1/2}$. Let $A_\alpha : T \to T$ with domain

$$\operatorname{dom}(A_\alpha) = \left\{ f \in T \,\middle|\, \begin{array}{c} f, D^\alpha f \in AC\big(\cup_0^{m-1}(d_i, d_{i+1})\big), \\ \ell_\alpha f \in L_2^\alpha(0,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\}$$

by

$$A_\alpha f = \ell_\alpha f, \quad f \in \operatorname{dom}(A_\alpha).$$

Suppose f and g are two solutions $\ell_\alpha f = \lambda f$, $\ell_\alpha g = \lambda g$ satisfying the jump conditions (2.3), the modified Wronskian

$$W_\alpha(f, g) = w(x)\big(f(x)D^\alpha g(x) - D^\alpha f(x)g(x)\big) \tag{2.4}$$

is constant for all $x \in [0, d_1) \cup_1^{m-2} (d_i, d_i + 1) \cup (d_{m-1}, \pi]$. Using the above formula $W_\alpha(f, g)(x) = W_\alpha(f, g)(x_0)$, for $x_0 \in [0, d) \cup (d, \pi]$. So, $W_\alpha(f, g)$ does not depend on x.

Lemma 2.4. *The operator A_α is self-adjoint on $L_2^\alpha((0,\pi); w)$.*

Let $\varphi(x,\lambda)$ and $v(x,\lambda)$ be the solutions of (2.1) with the following initial conditions

$$\varphi(0,\lambda) = 1, \quad D^\alpha\varphi(0,\lambda) = -h, \tag{2.5}$$

$$\psi(\pi,\lambda) = 1, \quad D^\alpha\psi(\pi,\lambda) = -H, \tag{2.6}$$

and the jump conditions (2.3), respectively. The characteristic function is defined by

$$\Delta(\lambda) := W_\alpha(\varphi(\lambda), \psi(\lambda)) = B_1(\psi(\lambda)) = -w(\pi)B_2(\varphi(\lambda)). \tag{2.7}$$

Theorem 2.5. *Let $\lambda = \rho^2$ and $\tau := |\mathrm{Im}\rho|$. For CSLP (2.1)–(2.3) as $|\lambda| \to \infty$, the asymptotic forms of solutions and the characteristic function formula* *are in the following forms:*

$$\varphi(x,\lambda)=\begin{cases}\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0\le x<d_1,\\ \alpha_1\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+\alpha_1'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_1<x<d_2,\\ \alpha_1\alpha_2\cos\rho\left(\frac{\rho}{\alpha}x^\alpha\right)+\alpha_1'\alpha_2\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+\alpha_1\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_2^\alpha)\right) & \\ \quad+\alpha_1'\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_2<x<d_3,\\ \quad\vdots & \\ \alpha_1\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+ & \\ \quad+\alpha_1'\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+\cdots & \\ \quad+\alpha_1\alpha_2\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_{m-1}^\alpha)\right)+ & \\ \quad+\alpha_1'\alpha_2'\alpha_3...\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)+\cdots & \\ \quad+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_j^\alpha-2d_k^\alpha)\right) & \\ \quad+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_j^\alpha+2d_k^\alpha-2d_s^\alpha)\right)+\cdots & \\ \quad+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\right) & \\ \quad+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_{m-1}<x\le\pi,\end{cases}\tag{2.8}$$

$$D^\alpha\varphi(x,\lambda)=\begin{cases}\rho\left[-\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right]+O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0\le x<d_1,\\ \rho\left[-\alpha_1\sin\left(\frac{\rho}{\alpha}x^\alpha\right)-\alpha_1'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)\right]+O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_1<x<d_2,\\ \rho\left[-\alpha_1\alpha_2\sin\rho\left(\frac{\rho}{\alpha}x^\alpha\right)-\alpha_1'\alpha_2\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)-\alpha_1\alpha_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_2^\alpha)\right)\right. & \\ \quad\left.-\alpha_1'\alpha_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)\right]+O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_2<x<d_3,\\ \quad\vdots & \\ \rho\left[-\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right. & \\ \quad-\alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+\cdots & \\ \quad-\alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_{m-1}^\alpha)\right)+ & \\ \quad-\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)+\cdots & \\ \quad-\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha+2d_j^\alpha-2d_k^\alpha)\right) & \\ \quad-\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha-2d_j^\alpha+2d_k^\alpha-2d_s^\alpha)\right)+\cdots & \\ \quad\left.+-\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\right)\right] & \\ \quad+O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_{m-1}<x\le\pi,\end{cases}\tag{2.9}$$

where

$$\alpha_i=\frac{a_i+b_i}{2}\quad \textit{and}\quad \alpha_i'=\frac{a_i-b_i}{2},\quad i=1,2,\dots,m-1.\tag{2.10}$$

Also the similar asymptotic form holds for the solution ψ. Moreover, we have

$$\begin{aligned}\Delta(\lambda) =&\rho w(\pi)\Big[\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}\pi^\alpha\right)+\alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha-2d_1^\alpha)\right)+\cdots\\ &+\alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^\alpha-2d_{m-1}^\alpha)\right)+\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha+2d_1^\alpha-2d_2^\alpha)\right)\\ &+\cdots+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha+2d_j^\alpha-2d_k^\alpha)\right)\\ &+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha-2d_j^\alpha+2d_k^\alpha-2d_s^\alpha)\right)+\cdots\\ &+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\right)\Big]\\ &+O\left(\exp\left(\frac{\tau}{\alpha}\pi^\alpha\right)\right).\end{aligned} \tag{2.11}$$

Using Theorem 2.5 and Definition 2.1, we find

$$\begin{aligned}|\varphi(x,\lambda)|&=O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right),\\ |D^\alpha\varphi(x,\lambda)|=|x^{1-\alpha}\varphi'(x,\lambda)|&=O\left(|\rho|\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right),\ 0\le x\le\pi\end{aligned} \tag{2.12}$$

By changing x to $\pi-x$ and using the jump conditions (2.3) and Definition 2.1, we calculate the asymptotic forms of $v(x,\lambda)$ and $D^\alpha v(x,\lambda)$. Specially,

$$\begin{aligned}|\psi(x,\lambda)|&=O\left(\exp\left(\frac{\tau}{\alpha}(\pi-x)^\alpha\right)\right),\\ |D^\alpha\psi(x,\lambda)|=|x^{1-\alpha}\psi'(x,\lambda)|&=O\left(|\rho|\exp\left(\frac{\tau}{\alpha}(\pi-x)^\alpha\right)\right),\quad 0\le x\le\pi.\end{aligned} \tag{2.13}$$

Theorem 2.6. *Let $\lambda_n=\rho_n^2$ be the eigenvalues of the problem L_α, then we have the following asymptotic formula*

$$\rho_n=\alpha\pi^{1-\alpha}n+O(1) \tag{2.14}$$

as $n\to\infty$.

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