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INVERSE PROBLEM FOR DIRAC OPERATOR WITH DISCONTINUOUS CONDITIONS

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Abstract. In this manuscript, we study the inverse problem for Dirac operators with discontinuity conditions inside an interval. It is shown that the potential functions can be uniquely determined by a part of a set of values of eigenfunctions at an interior point and parts of one or two sets of eigenvalues.

1. Introduction

Let us consider the Dirac operator

$$\ell[y(x)] := By'(x) + \Omega(x)y(x) = \lambda y(x) \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} U(y) &:= y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \\ V(y) &:= y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \end{aligned} \tag{1.2}$$

and the jump conditions

$$C(y) := y(d+0) = Ay(d-0), \tag{1.3}$$

where $x \in I := [0,d) \cup (d,\pi]$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$,

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$y(x) = \left(\; y_1(x), \; y_2(x) \; \right)^T$, and $A = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$.

In this paper, the functions $p(x)$ and $q(x)$ are real valued in $L_2(0, \pi)$, $a \in \mathbb{R} - \{0\}$, α, $\beta \in [0, \pi)$ and λ is a spectral parameter. For simplicity we use $L = L(\Omega(x); \alpha; \beta; d)$ for the above system of equation. It is easy to see that the operator L is a self-adjoint operator. Indeed the operator L has a discrete spectrum consisting simple and real eigenvalues λ_n, for $n \in \mathbb{Z}$.

In the paper [1], Amirov study the direct and inverse problems for Dirac operators with discontinuities inside an interval. Furthermore, direct or inverse spectral problems for Dirac operators were extensively studied in [5, 6], and the references therein. In this manuscript, we study the inverse problem for Dirac differential operators with discontinuity conditions. It is shown that the potential functions can be uniquely determined by a part of a set of values of eigenfunctions at an interior point and parts of one or two sets of eigenvalues.

2. Preliminaries

Let the functions $u(., \lambda) : I \to \mathbb{R}^2$ be

$$\begin{aligned} &Bu'(x) + \Omega(x)u(x) = \lambda u(x) \\ &u_1(0) = \sin\alpha, \;\; u_2(0) = -\cos\alpha. \end{aligned} \tag{2.1}$$

with the jump conditions (1.3) where $u(x, \lambda) = (u_1(x, \lambda), u_2(x, \lambda))^T$. It is shown in [2], [3] and [4] that there exit kernels $K(x, t) = (K_{ij}(x, t)^2_{i,j=1})$ with entire continuously differentiable on $0 \le t \le x < d$ such that the solution $u(x, \lambda)$ is

$$u(x, \lambda) = u_\circ(x, \lambda) + \int_0^x K(x, t) u_\circ(t, \lambda) dt \tag{2.2}$$

Here $u_\circ(x, \lambda) = \left(\; u_{\circ 1}(x, \lambda), \;\; u_{\circ 2}(x, \lambda) \; \right)^T$. It is easy to check that the following functions are solutions of (1.1) with $\Omega(x) = 0$,

$$u_{\circ 1}(x, \lambda) = \begin{cases} \sin(\lambda x + \alpha), & 0 \le x < d, \\ a^+ \sin(\lambda x + \alpha) + a^- \sin(\lambda(2d - x) + \alpha), & d < x \le \pi. \end{cases} \tag{2.3}$$

$$u_{\circ 2}(x, \lambda) = \begin{cases} -\cos(\lambda x + \alpha), & 0 \le x < d, \\ -a^+ \cos(\lambda x + \alpha) + a^- \cos(\lambda(2d - x) + \alpha), & d < x \le \pi. \end{cases} \tag{2.4}$$

where $a^+ = \frac{1}{2}\left(a + \frac{1}{a}\right)$, and $a^- = \frac{1}{2}\left(a - \frac{1}{a}\right)$. The characteristic function for $\left(\; u_{\circ 1}(x, \lambda), \;\; u_{\circ 2}(x, \lambda) \; \right)^T$ is

$$\Delta_\circ(\lambda) := a^+ \sin(\lambda\pi + \alpha - \beta) + a^- \sin(\lambda(2d - \pi) + \alpha + \beta). \tag{2.5}$$

The roots of the entire function $\Delta_\circ(\lambda)$ are simple and real. The roots of $\Delta_\circ(\lambda)$ is

$$\lambda_n^\circ = n + M_n \tag{2.6}$$

such that $\sup_n M_n < M < \infty$.

Suppose $v(x,\lambda) = \begin{pmatrix} v_1(x,\lambda), & v_2(x,\lambda) \end{pmatrix}^T$ be the solution of (1.1) with the initial conditions

$$v(\pi,\lambda) = \begin{pmatrix} \sin\beta, -\cos\beta \end{pmatrix}^T.$$

By changing x to $\pi - x$ one can obtain the similar form of (1.2) for the solution $v(x,\lambda)$ on the interval $(d,\pi]$.

We define the characteristic function for the operator L of the form

$$\Delta(\lambda) := \langle u(x,\lambda), v(x,\lambda)\rangle = \int_0^\pi (u_1\bar{v}_1 + u_2\bar{v}_2)dx$$

The characteristic function $\Delta(\lambda)$ is independent of x. It flows from (2.2) and the same form of (2.2) for $v(x,\lambda)$ on the jump point $x = d$, so we have

$$\Delta(\lambda) = \Delta_\circ(\lambda) + O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right) \tag{2.7}$$

where $\tau = |\mathrm{Im}\lambda|$. The zeros of $\Delta(\lambda)$ are the eigenvalues of L and hence it has only simple and real zeros λ_n. We denote by $y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T$, $n \in \mathbb{Z}$, the corresponding eigenfunctions.

Theorem 2.1. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \to \infty$:*

$$\lambda_n = n + O(1). \tag{2.8}$$

Suppose $v(x,\lambda) = \begin{pmatrix} v_1(x,\lambda), & v_2(x,\lambda) \end{pmatrix}^T$ be the solution of (1.1) with the initial conditions

$$v(\pi,\lambda) = \begin{pmatrix} \sin\beta, -\cos\beta \end{pmatrix}^T.$$

By changing x to $\pi - x$ one can obtain the similar form of (1.2) for the solution $v(x,\lambda)$ on the interval $(d,\pi]$. Define the characteristic function for the operator L of the form

$$\Delta(\lambda) := \langle u(x,\lambda), v(x,\lambda)\rangle.$$

The characteristic function $\Delta(\lambda)$ is independent of x. It flows from (2.2) and the same form of (2.2) for $v(x,\lambda)$ on the jump point $x = d$, so we have

$$\Delta(\lambda) = \Delta_\circ(\lambda) + O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right) \tag{2.9}$$

where $\tau = |\mathrm{Im}\lambda|$. The zeros of $\Delta(\lambda)$ are the eigenvalues of L and hence it has only simple and real zeros λ_n. We denote by $y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T$, $n \in \mathbb{Z}$, the corresponding eigenfunctions.

Theorem 2.2. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \to \infty$:*

$$\lambda_n = n + O(1). \tag{2.10}$$

3. INVERSE PROBLEM

Let us introduce a second Dirac operator $\tilde{L} = L(\tilde{\Omega}(x); \alpha; \beta; d)$ here

$$\tilde{\Omega}(x) = \begin{pmatrix} \tilde{p}(x) & \tilde{q}(x) \\ \tilde{q}(x) & -\tilde{p}(x) \end{pmatrix}$$

with a real valued function $\tilde{p}(x),\ \tilde{q}(x) \in L^2(0,\ \pi)$. The eigenvalues and the corresponding eigenfunctions of $\tilde{L}$ are denoted by $\tilde{\lambda}_n$ and $\tilde{y}_n(x) = (\tilde{y}_{n,1}(x), \tilde{y}_{n,2}(x))^T (n \in \mathbb{Z})$, respectively.

Theorem 3.1. *If*

$$\lambda_n = \tilde{\lambda}_n,\ \langle y_n, \tilde{y}_n \rangle_{d-0} = 0$$

for any $n \in \mathbb{Z}$ and $d \le \frac{\pi}{2}$ then $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ a.e on the $[0, d)$.

Remark 3.2. We can easily obtain if y, z be the solution of (1.1) and satisfy the jump conditions (1.3) and $\langle y, z \rangle_{(a-0)} = 0$ then $\langle y, z \rangle_{(a+0)} = 0$

Corollary 3.3. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point. Let $\lambda_n = \tilde{\lambda}_n$, and $\langle y_n, \tilde{y}_n \rangle_{(d-0)} = 0$, for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $(d, \pi]$.*

Remark 3.4. For $d = \frac{\pi}{2}$ from Theorems 3.1 and 3.3, we get $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.

Theorem 3.5. *Let $d \in (\frac{\pi}{2}, \pi]$ be a jump point and $\sigma > \frac{2a}{\pi} - 1$. Let*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)},\ \text{and } \langle y_n, \tilde{y}_n \rangle_{(d-0)} = 0,$$

for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, d) \cup (d, \pi]$.

Corollary 3.6. *Let $d \in (0, \frac{\pi}{2}]$ be a jump point, fix $b \in (0, d]$ and $\sigma_1 > \frac{2b}{\pi}$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ $\langle y_n, \tilde{y}_n \rangle_{d-0} = 0$, for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.*

Let $r(n)$ be a subsequence of natural numbers such that

$$r(n) = \frac{n}{\sigma_2}(1 + \epsilon_{2n}), 0 < \sigma_2 \le 1, \epsilon_{2n} \to 0 \tag{3.1}$$

Corollary 3.7. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point, fix $\sigma > \frac{2d}{\pi} - 1$ and $\sigma_2 > 2 - \frac{2d}{\pi}$. If for each $n \in \mathbb{N}$*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)},\ \langle y_{r(n)}, \tilde{y}_{r(n)} \rangle_{(d-0)} = 0,$$

then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.

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