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MULTIPLE SOLUTIONS FOR AN ANISOTROPIC VARIABLE EXPONENT PROBLEM WITH NEUMANN BOUNDARY CONDITION

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Abstract. By using variational methods and critical point theory, we establish the existence of multiple solutions for a Neumann problem. We prove the existence by applying the theory of variable exponent Sobolev spaces.

1. Introduction

In the present paper, we want to establish the existence of multiple solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + h(x) \sum_{i=1}^{N} a_i(x, u) = \lambda f(x, u) & in\ \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N(\ N \geq 3)$ with smooth boundary $\partial\Omega$, ν_i of the outer normal unit vector to $\partial\Omega$, λ is a positive parameter, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $h(x)$ is a positive function such that $h(.) \in L^\infty(\Omega)$ and

$$h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0, \tag{1.2}$$

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* Speaker.

and

$$h^{+}=\operatorname*{ess\,sup}_{x\in\Omega}h(x)>0. \tag{1.3}$$

In recent years variational problems with a nonstandard growth condition have attracted the interest of many specialists and have led to many related papers; for the generalized space theory, we refer the reader to [12], for the existence and multiplicity of solutions of elliptic equations with nonstandard growth condition, especially involving the $p(x)$-Laplacian, we refer the reader to [13]. For application background, we refer the reader to [20]. For anisotropic quasilinear elliptic equations, the working space is more general than for the usual $p(x)$-Laplacian, in that different space directions have different roles, so it possesses more inhomogeneity. In [9] Ding, Li and Bisci established the existence of three solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u=\lambda f(x,u) & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & x\in\partial\Omega, \end{cases} \tag{1.4}$$

They proved that this problem possesses at least three distinct weak solutions. We mention that if we focus on a certain type of the functions a_i as

$$a_i(x,s)=|s|^{p_i(x)-2}s,$$

for all $i\in\{1,...,N\}$ and $h(x)=1$, then our studying problem convert to the problem (1.4).
The Elliptic problems in anisotropic form concerning the Sobolev space with variable exponents have recently attracted the attention of many mathematicians; see [3, 6, 7, 10, 21].
In recent years, studying the existence of nontrivial solution for boundary value problems by applying Theorem 5.1 [2] have attracted the interest of many researchers, see for example [14, 15, 16].
In [4], the authors study a general class of anisotropic problems with variable exponents and constant Dirichlet condition

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}a_i(x,\partial_{x_i}u)+b(x)|u|^{p_M(x)-2}u=\lambda f(x,u) & x\in\Omega,\\ u(x)=constant & x\in\partial\Omega. \end{cases} \tag{1.5}$$

The plane of this article is as follows. In section 2 we introduce our notation and a suitable abstract setting. In section 3 we present the main results.

2. Preliminaries

In this section we recall some definition and the main properties of the spaces with variable exponents together with some results that we need for the proof of our main results.
Define

$$C_{+}(\Omega):=\{p\ :\ p\in C(\overline{\Omega})\text{ and }p(x)>1,\ \forall x\in\overline{\Omega}\}.$$

For $p \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{p(x)}(\Omega) = \{u \ : \ u \in S(\Omega), \ \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

where $S(\Omega)$ denotes the set of all measurable real functions on Ω.
This space, endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\tau > 0 \ : \ \int_\Omega |\frac{u(x)}{\tau}|^{p(x)} dx \leqslant 1\},$$

is a separable and reflexive Banach space. We refer to [8, 12, 18, 19] for the elementary properties of these spaces.

Proposition 2.1. *([12]) If $0 < |\Omega| < \infty$ and q_1, q_2 are variable exponents so that $q_1(x) \leqslant q_2(x)$ a. e. in Ω then the embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}$ is continuous.*

Let

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \qquad p^- = \min_{x \in \overline{\Omega}} p(x).$$

Proposition 2.2. *([8]) The conjugate space of $L^{p(x)}$ is $L^{p'(x)}$ where $p'(x)$ is the conjugate function of $p(x)$, i.e.*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Holder-type inequality

$$|\int_\Omega u(x)v(x)dx| \leqslant (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}.$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(x)}$, we set

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \ : \ |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}. \tag{2.1}$$

The space $W^{1,p(x)}(\Omega)$, equipped with the norm 2.1 becomes a separable, reflexive and uniformly convex Banach space. See for more details [1].
For $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_h = \inf\{\eta > 0 \ : \ \int_\Omega (|\frac{\nabla u}{\eta}|^{p(x)} + h(x)|\frac{u}{\eta}|^{p(x)})dx \leqslant 1\}. \tag{2.2}$$

Remark 2.3. According to [11], $\|u\|_h$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{W^{1,p(x)}(\Omega)}$.

Proposition 2.4. *([12]) For $p \in C(\overline{\Omega})$ such that $p^- > N$ for all $x \in \overline{\Omega}$, there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Proposition 2.5. *([12]) Let $\kappa(u)=\int_\Omega|\nabla u|^{p(x)}dx$. For $u_n,u\in W^{1,p(x)}(\Omega)$, we have*

(1) $\|u\|<(=;>)1 \iff \kappa(u)<(=;>)1$,
(2) $\|u\|\geqslant 1 \implies \|u\|^{p^-}\leqslant\kappa(u)\leqslant\|u\|^{p^+}$,
(3) $\|u\|\leqslant 1 \implies \|u\|^{p^+}\leqslant\kappa(u)\leqslant\|u\|^{p^-}$,
(4) $\|u_n\|\to 0 \iff \kappa(u_n)\to 0$, *and* $\|u_n\|\to\infty \iff \kappa(u_n)\to\infty$.

We assume in the sequel that Ω is a bounded open domain in $\mathbb{R}^N$ and we denote by

$$\overrightarrow{p}(.):\overline{\Omega}\to\mathbb{R}^N$$

the vectorial function

$$\overrightarrow{p}(.)=(p_1(.),...,p_N(.)).$$

We define $W^{1,\overrightarrow{p}(.)}(\Omega)$, the anisotropic variable exponent Sobolev space with respect to the norm

$$\|u\|_{\overrightarrow{p}(.)}=\|u\|_{W^{1,\overrightarrow{p}(.)}(\Omega)}=\sum_{i=1}^N\inf\{\sigma>0\,;\,(\int_\Omega|\frac{\partial_{x_i}u}{\sigma}|^{p_i(x)}dx+\int_\Omega h(x)|\frac{u}{\sigma}|^{p_i(x)}dx)\leqslant 1\}. \tag{2.3}$$

It was argued in [11] that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is a reflexive Banach space and a seperable space.
On the other hand, for the convenience of working with the space $W^{1,\overrightarrow{p}(.)}(\Omega)$ we introduce $\overrightarrow{p}_+$, $\overrightarrow{p}_-$ in $\mathbb{R}^N$ as

$$\overrightarrow{p}_+=(p_1^+,...,p_N^+),\qquad \overrightarrow{p}_-=(p_1^-,...,p_N^-),$$

and

$$p_+^+=\max\{p_1^+,...,p_N^+\},\qquad p_-^-=\min\{p_1^-,...,p_N^-\}.$$

Suppose that

$$\sum_{i=1}^N\frac{1}{p_i^+}<1. \tag{2.4}$$

Then it is proved in [9] that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c>0$ such that

$$\|u\|_\infty\leqslant c\|u\|_{\overrightarrow{p}(.)},\qquad\forall\ u\in W^{1,\overrightarrow{p}(.)}(\Omega), \tag{2.5}$$

where $\|u\|_\infty:=\sup_{x\in\overline{\Omega}}|u(x)|$.
In [2] Bonnano proposed the following innovative theorems for the study of nonlinear problems:

Theorem 2.6. *([2] Theorem 5.2) Let X be a reflexive real Banach space, $\Phi:X\to\mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits* *a continuous inverse on* X^*, *and* $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let* $I_\lambda = \Phi - \lambda\Psi$ *and for fix* $r > \inf_X \Phi$ *let* φ *be the function defined as*

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(u)}{r - \Phi(v)}.$$

Then, for each $\lambda \in]0, \dfrac{1}{\varphi(r)}[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]-\infty, r[)$ *such that* $I_\lambda(u_{(0,\lambda)}) \leqslant I_\lambda(u)$ *for all* $u \in \Phi^{-1}(] - \infty, r[)$ *and* $I'_\lambda(u_{(0,\lambda)}) = 0$.

Let us denote by $A_i : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$, $i \in \{1, ..., N\}$, and by $F : \Omega\times\mathbb{R} \to \mathbb{R}$ the antiderivatives of the Carathéodory functions $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, respectively $f : \Omega \times \mathbb{R} \to \mathbb{R}$; that is,

$$A_i(x, s) = \int_0^s a_i(x, t)dt, \tag{2.6}$$

$$F(x, s) = \int_0^s f(x, t)dt. \tag{2.7}$$

For every $i \in \{1, ..., N\}$, we work under the following assumptions:

(b_1) There exists a positive constant $\overline{c}_i$ such that a_i fulfills

$$|a_i(x, s)| \leqslant \overline{c}_i|s|^{p_i(x)-1}, \tag{2.8}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.
So

$$|A_i(x, s)| \leqslant \overline{c}_i|s|^{p_i(x)}. \tag{2.9}$$

(b_2) There exists $k_i > 0$ such that

$$k_i|s|^{p_i(x)} \leqslant a_i(x, s)s \leqslant p_i(x)A_i(x, s), \tag{2.10}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(b_3) $a_i(x, 0) = 0$ for all $x \in \partial\Omega$.

(b_4) a_i fulfills

$$(a_i(x, s) - a_i(x, t))(s - t) > 0, \tag{2.11}$$

for all $x \in \Omega$ and $s, t \in \mathbb{R}$ with $s \neq t$.

(b_5) There exist $k > 0$ and $q \in C_+(\overline{\Omega})$ with $p_+^+ < q^- < q^+ < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, where

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^N \dfrac{1}{p_i(x)}},$$

$$r^*(x) = \begin{cases} \dfrac{N\ r(x)}{N - r(x)} & if\ r(x) < N, \\ \infty & if\ r(x) \geqslant N, \end{cases} \tag{2.12}$$

such that f verifies

$$|f(x, s)| \leqslant k(1 + |s|^{q(x)-1}), \tag{2.13}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(b_6) There exist $\gamma > p_+^+$ and $s_0 > 0$ such that the Ambrosetti-Rabinowitz condition
$$0 < \gamma F(x,s) < sf(x,s),$$
holds for all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| > s_0$.

Definition 2.7. We say that $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ is a weak solution of the problem (1.1) if

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\ \partial_{x_i} v\ dx + \int_\Omega \sum_{i=1}^N h(x) a_i(x,u)\ v\ dx = \lambda \int_\Omega f(x,u)\ v\ dx$$

for all $v \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

Foe each $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$, let the functionals $\Phi, \Psi : W^{1,\overrightarrow{p}(.)}(\Omega) \to \mathbb{R}$ be defined by

$$\Phi(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\ dx + \int_\Omega \sum_{i=1}^N h(x) A_i(x,u)\ dx, \tag{2.14}$$

and

$$\Psi(u) = \int_\Omega F(x, u(x))\ dx. \tag{2.15}$$

By standard arguments, it follows that the functionals Φ and Ψ are well defined and of class C^1, and with the derivatives given by

$$< \Phi'(u), v >= \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\ \partial_{x_i} v\ dx + \int_\Omega \sum_{i=1}^N h(x) a_i(x,u)\ v\ dx, \tag{2.16}$$

$$< \Psi'(u), v >= \int_\Omega f(x, u(x))\ v(x)\ dx, \tag{2.17}$$

for any $u, v \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

Lemma 2.8. *For all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ we have*

(*i*) *if $\|u\|_{\overrightarrow{p}(.)} \geqslant 1$ then*
$$\frac{\min\{k_1, ..., k_N\}}{p_+^+} \|u\|_{\overrightarrow{p}(.)}^{p_-^-} \leqslant \Phi(u) \leqslant \max\{\bar{c}_1, ..., \bar{c}_N\} \|u\|_{\overrightarrow{p}(.)}^{p_+^+},$$

(*ii*) *if $\|u\|_{\overrightarrow{p}(.)} \leqslant 1$ then*
$$\frac{\min\{k_1, ..., k_N\}}{p_+^+} \|u\|_{\overrightarrow{p}(.)}^{p_+^+} \leqslant \Phi(u) \leqslant \max\{\bar{c}_1, ..., \bar{c}_N\} \|u\|_{\overrightarrow{p}(.)}^{p_-^-}.$$

Proof. It is an immediate result of (2.8) and (2.13) in the case where k_i corresponds to p_i. □

Lemma 2.9. *The functional Φ is coercive.*

Proof. Let $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ be such that $\|u\|_{\overrightarrow{p}(.)} \to \infty$. By Lemma (2.8) we deduce that for any $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(.)} > 1$ we have

$$\Phi(u) \geqslant \frac{\min\{k_1, ..., k_N\}}{p_+^+} \|u\|_{\overrightarrow{p}(.)}^{p_-^-}.$$

Hence Φ is coercive. □

The following theorem guarantees the coercivity and continuity of the Gâteaux derivative of the functional Φ.

Theorem 2.10. *([17] Theorem 6.2.1). Let X be a reflexive Banach space, and let $f : M \subseteq X \to \mathbb{R}$ be Gâteaux differentiable over the closed, convex set M. Then the following conditions are equivalent:*

(i) f is convex over M.

(ii) We have

$$f(u) - f(v) \geqslant < f'(v), u - v >_{X^* \times X} \qquad \forall u, v \in M,$$

where X^* denotes the dual of the space X.

(iii) The first Gâteaux derivative is monotone, that is,

$$< f'(u) - f'(v), u - v >_{X^* \times X} \geqslant 0 \qquad \forall u, v \in M.$$

(iv) The second Gâteaux derivative of f exists and it is positive, that is,

$$< f''(u) \circ v, v >_{X^* \times X} \geqslant 0 \qquad \forall v \in M.$$

Lemma 2.11. *The functional Φ is sequentially weakly lower semi-continuous.*

Proof. By ([5] Section 1.4), it is enough to prove that Φ is lower semi-continuous. To this end, fix $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ and $\varepsilon > 0$. By (b_4) and Theorem 2.10 (iii) we deduce that for any $v \in W^{1,\overrightarrow{p}(.)}(\Omega)$, the following inequality holds:

$$\Phi(v) - \Phi(u) \geqslant < \Phi'(u), v - u >_{W^{1,\overrightarrow{p}(.)}(\Omega) \times (W^{1,\overrightarrow{p}(.)}(\Omega))^*},$$

$$\Phi(v) \geqslant \Phi(u) + \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)(\partial_{x_i} v - \partial_{x_i} u) dx + \int_\Omega \sum_{i=1}^N h(x) a_i(x, u)(v - u) dx,$$

using (b_1) and

$$\Phi(v) \geqslant \Phi(u) - \max\{\overline{c}_1, ..., \overline{c}_N\} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-1} \ |\frac{\partial}{\partial_{x_i}}(v - u)| \ dx$$

$$- \|h\|_{L^\infty} \max\{\overline{c}_1, ..., \overline{c}_N\} \int_\Omega \sum_{i=1}^N |u|^{p_i(x)-1} \ |v - u| \ dx,$$

$$\geqslant \Phi(u) - (\frac{1}{p_-^-} + \frac{1}{p'^-_-}) \max\{\overline{c}_1, ..., \overline{c}_N\} \int_\Omega \sum_{i=1}^N | \ |\partial_{x_i} u|^{p_i(x)-1}|_{L^{p'_i(x)}(\Omega)} \ |\partial_{x_i} v - \partial_{x_i} u|_{L^{p_i(x)}(\Omega)} \ dx$$

$$-\|h\|_{L^\infty}(\frac{1}{p_-^-}+\frac{1}{p_-'^-})\max\{\overline{c}_1,...,\overline{c}_N\}\int_\Omega\sum_{i=1}^N\big|\,|u|^{p_i(x)-1}\big|_{L^{p_i'(x)}(\Omega)}\,|v-u|_{L^{p_i(x)}(\Omega)}\,dx.$$

The above inequality and relation (4) imply that there exists $M>0$ such that

$$\Phi(v)\geqslant\Phi(u)-M\|v-u\|\geqslant\Phi(u)-\varepsilon,$$

for all $v\in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\|v-u\|<\delta=\dfrac{\varepsilon}{M}$. Therefore Φ is sequentially weakly lower semi-continuous. □

Now, we state our first main results as follows.

Theorem 2.12. *Assume that*

$$\sup_{\gamma>0}\frac{\min\{k_1,...,k_n\}\gamma^{p_-^-}}{\int_\Omega\sup_{|t|\leqslant\gamma}F(x,t)dx}>p_+^+c^{p_-^-},\tag{2.18}$$

where c is the constant defined in (2.5). Then the problem (1.1) admits at least one weak solution in $W^{1,\overrightarrow{p}(.)}(\Omega)$.

Proof. Our aim is to apply Theorem 2.6 to our problem. To this end, let Φ, Ψ be the functionals defined in (2.14) and (2.15). $\Psi':W^{1,\overrightarrow{p}(.)}(\Omega)\to W^{1,\overrightarrow{p}(.)}(\Omega)^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on $W^{1,\overrightarrow{p}(.)}(\Omega)$. Fr fixed $u\in W^{1,\overrightarrow{p}(.)}(\Omega)$, let $u_n\to u$ weakly in $W^{1,\overrightarrow{p}(.)}(\Omega)$ as $n\to\infty$, then u_n converges uniformly to u on Ω as $n\to\infty$(see [22]). Since f is continuous in $\mathbb{R}$ for every $W^{1,\overrightarrow{p}(.)}(\Omega)\in\Omega$ so $f(x,u_n)\to f(x,u)$, as $n\to\infty$. Hence, $\Psi'(u_n)\to\Psi'(u)$ as $n\to\infty$. Thus the functional Ψ' is strongly continuous on $W^{1,\overrightarrow{p}(.)}(\Omega)$, which implies that Ψ' is a compact operator by proposition 26.2 of [22] (On the other hand the fact that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $\Psi':W^{1,\overrightarrow{p}(.)}(\Omega)\to W^{1,\overrightarrow{p}(.)}(\Omega)^*$ is compact).
Now, we observe that Φ' is uniformly monotone. It follows that the functional
$\Phi':W^{1,\overrightarrow{p}(.)}(\Omega)\to(W^{1,\overrightarrow{p}(.)}(\Omega))^*$ has a continuous inverse operator on $(W^{1,\overrightarrow{p}(.)}(\Omega))^*$, where $(W^{1,\overrightarrow{p}(.)}(\Omega))^*$ denotes the dual space of $W^{1,\overrightarrow{p}(.)}(\Omega)$. Furthermore, according to Lemma 2.9 and Lemma 2.11 Φ is coercive and sequentially weakly lower semicontinuous.
So the functionals Φ, Ψ satisfy all regularity assumptions requested in Theorem 2.6.
By using condition (2.18), there exists $\overline{\gamma}>0$ such that

$$\frac{\min\{k_1,...,k_n\}\overline{\gamma}^{p_-^-}}{\int_\Omega\sup_{|t|\leqslant\overline{\gamma}}F(x,t)dx}>p_+^+c^{p_-^-}.\tag{2.19}$$

Choose

$$r=\frac{\min\{k_1,...,k_n\}}{p_+^+}(\frac{\overline{\gamma}}{c})^{p_-^-}.$$

Moreover, for all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\Phi(u) < r$, from lem 2.8, one has

$$\|u\|^p \leqslant \max\{(p_+^+ r)^{\frac{1}{p_+^+}}, (p_+^+ r)^{\frac{1}{p_-^-}}\}.$$

So, due to the (2.5), one has $\|u\|_\infty < \overline{\gamma}$.
From the definition of r, it follows that

$$\Phi^{-1}(-\infty, r) = \{u \in W^{1,\overrightarrow{p}(.)}(\Omega); \Phi(u) < r\} \subseteq \{u \in W^{1,\overrightarrow{p}(.)}(\Omega); |u| \leqslant \overline{\gamma}\},$$

and this follows

$$\Psi(u) \leqslant \sup_{u \in \Phi^{-1}(-\infty, r)} \int_\Omega F(x, u(x))dx \leqslant \int_\Omega \sup_{|t| \leqslant \overline{\gamma}} F(x,t)dx,$$

for every $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u)<r} \Psi(u) \leqslant \int_\Omega \sup_{|t|\leqslant\overline{\gamma}} F(x,t)dx.$$

From the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\varphi(r) = \inf_{v \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(u)}{r - \Phi(v)} \leqslant \frac{\sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u)}{r}$$

$$\leqslant \frac{\int_\Omega \sup_{|t|\leqslant\overline{\gamma}} F(x,t)dx}{\frac{\min\{k_1, ..., k_n\}}{p_+^+}(\frac{\overline{\gamma}}{c})^{p_-^-}}.$$

At this point, we see that

$$\varphi(r) \leqslant \frac{\int_\Omega \sup_{|t|\leqslant\overline{\gamma}} F(x,t)dx}{\frac{\min\{k_1, ..., k_n\}}{p_+^+}(\frac{\overline{\gamma}}{c})^{p_-^-}}. \tag{2.20}$$

From (2.18) and (2.19) one has $\varphi(r) < 1$. Hence, since $1 \in (0, \frac{1}{\varphi(r)})$, by applying Theorem2.6 the functional I_λ admits at least one critical point (local minima) $\overline{u} \in \Phi^{-1}(-\infty, r)$. □

Example 2.13. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leqslant 9\}$. consider the following problem

$$\begin{cases} -\sum_{i=1}^3 \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) + \sum_{i=1}^3 |u|^{p_i(x)-2}u = \lambda e^u(u + u^2) & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega, \end{cases} \tag{2.21}$$

where $p_i(x) = x_i^2 + 9$ for all $x_1, x_2, x_3 \in \mathbb{R}$. We have $F(t) = e^t(t^2 - t + 1) - 1$, for every $t \in \mathbb{R}$. We obtain $p_-^- = 9$ and $p_+^+ = 18$. Since

$$\sup_{\gamma>0} \frac{\gamma^9}{\sup_{|t|\leqslant\gamma} e^t(t^2 - t + 1) - 1} > 18c^9 meas(\Omega),$$

hence, Theorem 2.12 implies that the problem (2.21) admits at least one weak solution in $W^{1,\overrightarrow{p}(.)}(\Omega)$.

Now, we state second main results to find three weak solutions for the problem (1.1). Our approach is the following problem:

Theorem 2.14 ([2, Theorem 7.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Assume that there is $r \in]inf_X \Phi, sup_X \Psi[$ such that*

$$\varphi(r) < \rho(r),$$

where

$$\varphi(r) := \inf_{v \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)},$$

and

$$\rho(r) := \sup_{v \in \Phi^{-1}(]r, \infty[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{\Phi(v) - r}.$$

and for each $\lambda \in \left] \frac{1}{\rho(r)}, \frac{1}{\varphi(r)} \right[$ the function $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.
Then, for each $\lambda \in \left] \frac{1}{\rho(r)}, \frac{1}{\varphi(r)} \right[$ the function I_λ admits at least three critical points.

Remark 2.15 ([2]). If we assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{u} \in X$, with $\Phi(\overline{u}) > r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})},$$

then one has $\varphi(r) < \rho(r)$ and, in addition,

$$\left] \frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[.$$

Proposition 2.16 ([2, Proposition 2.2])**.** *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that the function $\Phi - \Psi$ is coercive.*
Then, for all $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$, the function $\Phi - \Psi$ satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Theorem 2.17. *Assume that c be a positive constants with*

$$\frac{\int_\Omega \sup_{|t| \le c} F(x, t) dx}{r} < \frac{\int_\Omega F(x, \delta) dx}{\zeta c^{p_+^+}}, \tag{2.22}$$

and

$$0 < r < h^{-} \min\{k_1, \dots, k_N\} \frac{\delta^{p_-^-}}{p_+^+} meas(\Omega) \tag{2.23}$$

Then, for each parameter λ belonging to

$$\Lambda_{(r,\delta)} := \tag{2.24}$$

$$\left] \frac{\zeta c^{p_+^+}}{\int_\Omega F(x,\delta)dx}, \frac{r}{\int_\Omega \sup_{|t|\le c} F(x,t)dt} \right[,$$

the problem (1.1) *possesses at least three distinct weak solutions in* $W^{1,\overrightarrow{p}(.)}(\Omega)$.

Proof. Our aim is to apply Theorem 2.14 to our problem. To this end, let Φ, Ψ be the functionals defined in (2.15), (2.16).
Then $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to \left(W^{1,\overrightarrow{p}(.)}(\Omega)\right)^*$ is a compact operator. On the other hand the fact that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $\Psi' : W^{1,\overrightarrow{p}(.)}(\Omega) \to \left(W^{1,\overrightarrow{p}(.)}(\Omega)\right)^*$ is compact. Furthermore, according to Lemma 2.8 and 2.9, Φ is bounded from below and $\Phi - \lambda\Psi$ is coercive.
So the functionals Φ, Ψ satisfy in all regularity assumptions requested in Theorem 2.14, (we apply Proposition 2.16 and do not require (PS)-condition).
Here and in the sequel we have $\Phi(0) = \Psi(0) = 0$ and $\Phi(u) \ge 0$ for every $u \in X$. In the following, our aim is to verify condition (2.22). Put $\overline{v} := \delta \in W^{1,\overrightarrow{p}(.)}(\Omega)$ with $\delta > 1$, we have

$$h^{-} \min\{k_1, \dots, k_N\} \frac{\delta^{p_-^-}}{p_+^+} meas(\Omega) \le \Phi(\overline{v}) \le \zeta,$$

and

$$\Psi(\overline{v}) = \int_\Omega F(x,\delta)dx.$$

So, $r < \Phi(\overline{v})$. Moreover, for all $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty, r[)$, taking (2.6) into account, one has $|u(x)| < c$ for all $x \in \Omega$, from which it follows

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_\Omega F(x,u(x))dx \le \int_\Omega \sup_{|t|\le c} F(x,t)dx,$$

and

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \le \frac{\int_\Omega \sup_{|t|\le c} F(x,t)dx}{r}. \tag{2.25}$$

Moreover, one has

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} \ge \frac{\int_\Omega F(x,\delta)dx}{\zeta c^{p_+^+}}. \tag{2.26}$$

From (2.22) it follows that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}. \tag{2.27}$$

Now, we observe that

$$\begin{aligned}\varphi(r) &= \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)-\Psi(v)}{r-\Phi(v)}\\ &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r},\end{aligned}$$

and

$$\begin{aligned}\rho(r) &= \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{\Phi(v)-r}\\ &\geq \frac{\Psi(\overline{v})-\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{\Phi(\overline{v})-r}\\ &\geq \frac{\Psi(\overline{v})-r\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}}{\Phi(\overline{v})-r}\\ &= \frac{\Psi(\overline{v})}{\Phi(\overline{v})}.\end{aligned}$$

Hence, $\varphi(r) \leq \dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \dfrac{\Psi(\overline{v})}{\Phi(\overline{v})} \leq \rho(r)$.
So, all conditions that we need is verified. Since all the assumptions of Theorem 2.14 are satisfied, then, for each

$$\lambda \in \Lambda_{(r,\delta)} = \left]\frac{\zeta c^{p_+^+}}{\int_\Omega F(x,\delta)dx}, \frac{r}{\int_\Omega \sup_{|t|\leq c} F(x,t)dt}\right[,$$

the functional I_λ has at least three distinct critical points that are weak solutions of the problem(1.1). The proof is complete. □

Example 2.18. Define $g:\mathbb{R}\longrightarrow\mathbb{R}$ as follows

$$g(t) := \begin{cases} 0 & t<0,\\ t^{p_+^+} & 0\leq t\leq 1,\\ t^{\xi(x)} & t>1,\end{cases}$$

where $\xi(x)\in\left]0, p_-^- - 1\right[$. Further, let $m:\Omega\longrightarrow\mathbb{R}$ be a bounded measurable and positive function. From theorem 2.17, for each

$$\lambda > \frac{meas(\Omega)}{\|m\|_{L^1(\Omega)}} \inf_{\delta>0, G(\delta)>0} \frac{|\delta|^{p_+^+}}{p_-^- G(\delta)},$$

where $G(\delta) := \int_0^\delta g(t)dt$, the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) + \sum_{i=1}^{N} |u|^{p_i(x)-2}u = \lambda m(x)g(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

possesses at least three weak solutions in $W^{1,\overrightarrow{p}(.)}(\Omega)$.

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