SOME FIXED POINT THEOREMS FOR Γ-WARDOWSKI-GERAGHTY CONTRACTIONS IN Γ-B-METRIC SPACES

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Abstract. Motivated by Wardowski [Fixed Point Theory Appl. 2012:94, 2012] we introduce and study a new contraction and a new generalized b-metric space called α-β-F-Γ contraction and Γ-b-metric space respectively to prove a fixed point result as a generalization of the Banach contraction principle. Moreover, we discuss some illustrative contractions to highlight the realized improvements.

1. Introduction

After Bakhtin [4] and Czerwik [5, 6] introduced b-metric spaces, many authors attempted to generalize this practical concept. Kamran and others [7] presented controlled b-metric spaces. Following that, double-controlled b-metric spaces were introduced. Generalized b-metric spaces were also introduced by Parvaneh and Ghoncheh [9]. They used a function instead of the coefficient s, which was always above the half-tone of the first and third

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quarters and did not coincide with its inverse except at zero. This is a major limitation. In this article, by removing this limitation, we present a new category of generalized b-metric spaces, which will certainly be of interest to many researchers in the field of fixed point theory.

The most well-known conclusion in fixed point theory, the Banach contractive principle or Banach fixed point theorem, states that every contractive mapping in a complete metric space has a unique fixed point. By applying various types of contractive conditions in different spaces, this result can be generalized in a huge number of ways. A fixed point conclusion was demonstrated recently as a generalization of the Banach contraction principle by Wardowski [1], who also established a new contraction known as the F-contraction.

One of the interesting results which also generalizes the Banach contraction principle was given by Samet *et al.* [2] by defining α-ψ-contractive and α-admissible mappings.

Definition 1.1. [2] Let T be a self-mapping on a set X and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2. [3] Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(fx, fy) \geq 1, \quad x, y \in X;$

(T2) $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \quad \text{implies} \quad \alpha(x, y) \geq 1, \quad x, y, z \in X.$

Lemma 1.3. [3] *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Definition 1.4. [5] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$,

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Substituting the coefficient s by a function $\Omega : [0, \infty) \to [0, \infty)$ with some constraints, a two-variable function $\alpha : X \times X \to [0, \infty)$ and two two-variable functions $\alpha_1, \alpha_2 : X \times X \to [0, \infty)$ we have extended b-metric spaces, controlled b-metric spaces and double controlled b-metric spaces, respectively.

Removing some constraints on function Ω we have the following extension of the concept of b-metric space.

Definition 1.5. Let X be a (nonempty) set and $\Gamma : [0,\infty) \to [0,\infty)$ be a continuous increasing mapping. A function $d : X \times X \to [0,\infty)$ is a Γ-b-metric if, for all $x, y, z \in X$,

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $\Gamma[d(x,z)] \leq s \cdot \Gamma[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a Γ-b-metric space.

Note that a b-metric need not be a continuous function. So, we have this fact about Γ-b-metric spaces.

Example 1.6. Taking $\Gamma(x) = \operatorname{arcsinh} x$, the Γ-triangle inequality will be as follows:

$$\sinh^{-1}[d(x,z)] \leq s \cdot \sinh^{-1}[d(x,y) + d(y,z)].$$

So,

$$\begin{aligned}[d(x,z)] &\leq \sinh[s \cdot \sinh^{-1}[d(x,y) + d(y,z)]] \\ &= \frac{e^{2s\cdot \ln[[d(x,y)+d(y,z)]+\sqrt{[d(x,y)+d(y,z)]^2+1}]} - 1}{2e^{s\cdot \ln[[d(x,y)+d(y,z)]+\sqrt{[d(x,y)+d(y,z)]^2+1}]}} \\ &= \frac{\left[[d(x,y)+d(y,z)] + \sqrt{[d(x,y)+d(y,z)]^2+1}\right]^{2s} - 1}{2\left[[d(x,y)+d(y,z)] + \sqrt{[d(x,y)+d(y,z)]^2+1}\right]^{s}}.\end{aligned}$$

Lemma 1.7. *Let (X,d) be a Γ-b-metric, and suppose that $\{x_n\}$ and $\{y_n\}$ are Γ-b-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}\Gamma[d(x,y)] \leq \Gamma[\liminf_{n\to\infty} d(x_n,y_n)] \leq \Gamma[\limsup_{n\to\infty} d(x_n,y_n)] \leq s^2 \cdot \Gamma d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}\Gamma[d(x,z)] \leq \liminf_{n\to\infty} \Gamma[d(x_n,z)] \leq \limsup_{n\to\infty} \Gamma[d(x_n,z)] \leq s \cdot \Gamma[d(x,z)].$$

In this paper, we introduce the concept of α-β-F-Γ-contractions and obtain some fixed point results in Γ-b-metric spaces. Our results extend those of Wardowski and several other authors.

2. Fixed point results for α-admissible β-FG-contractions

Let $s > 1$ be a fixed real number. We will consider the following classes of functions.

Δ_F will denote the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(Δ_1) F is continuous and strictly increasing;

(Δ_2) for each sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$.

$\Delta_{F,\beta}$ will denote the set of pairs (F,β), where $F : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0,\infty) \to [0,1)$, such that

(Δ_3) for each sequence $\{t_n\} \subseteq R^+$, $\limsup_{n\to\infty} F(t_n) \geq 0$ if and only if $\limsup_{n\to\infty} t_n \geq 1$.

(Δ_4) for each sequence $\{t_n\} \subseteq [0,\infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$;

(Δ_5) for each sequence $\{t_n\} \subseteq R^+$, $\sum_{n=1}^{\infty} F(\beta(t_n)) = -\infty$;

Definition 2.1. Let (X,d) be a Γ-b-metric space and let T be a self-mapping on X. Also suppose that $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is an α-β-F-Γ-contraction if for all $x, y \in X$ with $1 \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$ we have

$$F\big(s\Gamma[d(Tx,Ty)]\big) \leq F\big(\Gamma[M_{s,\Gamma}(x,y)]\big) + F\big(\beta(M_{s,\Gamma}(x,y))\big), \tag{2.1}$$

where $F \in \Delta_F$, $(F,\beta) \in \Delta_{F,\beta}$ and

$$M_{s,\Gamma}(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{\Gamma^{-1}\big[\frac{d(x,Ty)+d(y,Tx)}{s}\big]}{2}\right\}. \tag{2.2}$$

Now we state and prove our main result of this section.

Theorem 2.2. *Let (X,d) be a complete Γ-b-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a triangular α-admissible mapping;*
(ii) *T is an α-β-F-Γ-contraction;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iv) *T is α-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point if $\alpha(x,y) \geq 1$ for all $x, y \in Fix(T)$.

Taking $F(t) = \ln t$, $\beta(x) = k$ where $k \in (0,1)$, and putting $\Gamma(x) = x^2$ in the above theorem, we obtain a generalization of the Banach contraction principle in the setup of Γ-b-metric spaces.

Corollary 2.3. *Let (X,d) be a complete Γ-b-metric space. Let T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ be a function such that the mapping T satisfies the following assertions:*

(i) *T is a triangular α-admissible mapping;*
(ii)
$$d(Tx,Ty) < \sqrt{k/s}\big(M_{s,\Gamma}(x,y)\big),$$
for all $x, y \in X$ with $1 \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$;
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iv) *T is α-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

In the following theorem we replace the α-continuity of T by another condition (iv$'$).

Theorem 2.4. *Let (X, d) be a complete Γ-b-metric space. Let $T : X \to X$ be a self-mapping and suppose that $\alpha : X \times X \to [0, \infty)$ be a function such that:*

(i) *T is a rectangular α-admissible mapping;*
(ii) *T is an α-β-F-Γ-contraction;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iv') *if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$, then $1 \leq \alpha(Tx_n, Tx)$ holds for all $n \in \mathbb{N}$.*

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

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