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MINIMAL INVARIANT PAIRS OF CYCLIC (NONCYCLIC) RELATIVELY NONEXPANSIVE MAPPINGS

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Abstract. We consider the class of cyclic (noncyclic) relatively nonexpansive mappings and study the structure of minimal invariant pairs in (strictly convex) Banach spaces. Then we conclude a well-known best proximity point (pair) theorem for such class of mappings.

1. Introduction

Let X be a Banach space and $C \subseteq X$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well known that if C is a nonempty, compact and convex subset of a Banach space X, then any nonexpansive mapping of C into C has a fixed point.

In the case that C is weakly compact and convex subset of a Banach space X, then the nonexpansive mapping T may not have a fixed point. If C possesses *normal structure*, then the existence of a fixed point is guaranteed by *Kirk's fixed point theorem* ([5]). We mention that every bounded, closed and convex subset of a *uniformly convex Banach space* X has normal structure.

Let A and B be two nonempty subsets of a normed linear space X. A mapping $T\colon A \cup B \to A \cup B$ is said to be *cyclic (noncyclic)* provided that $T(A) \subseteq B, T(B) \subseteq A$ $(T(A) \subseteq A, T(B) \subseteq B)$.

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In the case that T is cyclic a point $x^\star \in A \cup B$ is called a *best proximity point* of T whenever

$$\|x^\star - Tx^\star\| = \operatorname{dist}(A,B) := \inf\{\|x-y\| : x \in A,\ y \in B\}.$$

Moreover, a point $(p,q) \in A \times B$ is said to be a *best proximity pair* of the noncyclic mapping T if

$$p = Tp, \quad q = Tq, \quad \|p-q\| = \operatorname{dist}(A,B).$$

A mapping $T\colon A \cup B \to A \cup B$ is said to be *relatively nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|$ whenever $x \in A$ and $y \in B$.

To describe our results, we need some definitions and notations. We shall say that a pair (A,B) of subsets of a Banach space X satisfies a property if both A and B satisfy that property. For example, (A,B) is convex if and only if both A and B are convex; $(A,B) \subseteq (C,D) \Leftrightarrow A \subseteq C,$ and $B \subseteq D$. We shall also adopt the notation

$$\delta_x(A) = \sup\{d(x,y) \colon y \in A\} \text{ for all } x \in X,$$
$$\delta(A,B) = \sup\{\delta_x(B) : x \in A\}.$$

The *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$.

Definition 1.1. A Banach space X is said to be
(i) uniformly convex if there exists a strictly increasing function $\delta : (0,2] \to [0,1]$ such that the following implication holds for all $x,y,p \in X, R > 0$ and $r \in [0,2R]$:

$$\begin{cases} \|x-p\| \leq R, \\ \|y-p\| \leq R, \\ \|x-y\| \geq r \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| \leq (1 - \delta(\frac{r}{R}))R;$$

(ii) strictly convex if the following implication holds for all $x,y,p \in X$ and $R > 0$:

$$\begin{cases} \|x-p\| \leq R, \\ \|y-p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| < R.$$

Lemma 1.2. *([1]) Let (K_1,K_2) be a pair of nonempty subsets of a normed linear space X. Then*

$$\delta(K_1,K_2) = \delta(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2)).$$

Given (A,B) a pair of nonempty subsets of a Banach space, then its proximal pair is the pair (A_0,B_0) given by

$$A_0 = \{x \in A \colon \|x-y'\| = \operatorname{dist}(A,B) \text{ for some } y' \in B\},$$
$$B_0 = \{y \in B \colon \|x'-y\| = \operatorname{dist}(A,B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty weakly compact and convex then (A_0,B_0) is a nonempty weakly compact convex pair in X.

Definition 1.3. Let (A, B) be a nonempty pair in a Banach space X. Then (A, B) is said to be a proximinal pair if $A = A_0$ and $B = B_0$.

Definition 1.4. Let (A, B) be a nonempty pair of sets in a Banach space X. A point p in A (q in B) is said to be a diametral point with respect to B (w.r.t. A) if $\delta_p(B) = \delta(A, B)$ ($\delta_q(A) = \delta(A, B)$). A pair (p, q) in $A \times B$ is diametral if both points p and q are diametral.

2. Minimal invariant pairs of relatively nonexpansive mappings

The following lemmas play important roles in our coming discussions.

Lemma 2.1. [2] *Let (A, B) be a nonempty weakly compact convex pair of a Banach space X and let $T\colon A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then there exists $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty closed convex and T-invariant pair of subsets of (A, B) such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

Moreover, the pair (K_1, K_2) is proximinal.

Notation. Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and suppose $T : A \cup B \to A \cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping. By $\mathcal{M}_T(A, B)$ we denote the set of all nonempty, closed, convex, minimal and T-invariant pair $(K_1, K_2) \subseteq (A, B)$ such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$.

Lemma 2.2. (Lemma 3.1 of [3]) *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X and $T\colon A \cup B \to A \cup B$ a cyclic relatively nonexpansive mapping. If $(K_1, K_2) \in \mathcal{M}_T(A, B)$, then each pair $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ contains a diametral point (with respect to (K_1, K_2)).*

Lemma 2.3. (Lemma 3.8 of [3]) *Let (A, B) be a nonempty weakly compact convex pair of a strictly convex Banach space X. Let $T\colon A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If $(K_1, K_2) \in \mathcal{M}_T(A, B)$, then each $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ is a diametral pair (with respect to (K_1, K_2)), that is,*

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1, K_2).$$

3. Best proximity points (pairs)

Definition 3.1. Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. We say that the pair (A, B) has the H-property, if for any $(K_1, K_2) \in \mathcal{M}_T$,

$$\max\{\operatorname{diam}(K_1), \operatorname{diam}(K_2)\} \leq \delta(K_1, K_2).$$

It was announced in [4] that if (A, B) is a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T : A\cup B \to A\cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping, then (A, B) has the H-property.

Definition 3.2. Suppose (A, B) is a nonempty, disjoint, weakly compact and convex pair in a Banach space X and $T : A \cup B \to A \cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping such that (A, B) has the H-property. Define

$$\omega_T := \inf\{\frac{\max\{\mathrm{diam}(K_1), \mathrm{diam}(K_2)\}}{\delta(K_1, K_2)} : (K_1, K_2) \in \mathcal{M}_T\}.$$

It is clear that $\omega_T \in [0, 1]$.

Proposition 3.3. ([4]) *Let (A, B) be a nonempty, disjoint, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then $\omega_T = 0$.*

Theorem 3.4. ([4]) *Let (A, B) be a nonempty, disjoint, weakly compact and convex pair of subsets of a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. If $\omega_T = 0$, then has a best proximity point (pair).*

Corollary 3.5. ([2]) *Let (A, B) be a nonempty, disjoint, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then T has a best proximity point (pair).*

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