The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

ON THE EXISTENCE AND UNIQUENESS OF PSEUDO ALMOST AUTOMORPHIC SOLUTIONS FOR INTEGRO DIFFERENTIAL EQUATIONS WITH REFLECTION

ZAHRA EIDINEJAD*, REZA SAADATI*

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
zahra_eidinejad@mathdep.iust.ac.ir; rsaadati@eml.cc; rsaadati@iust.ac.ir

Abstract. In this paper we apply fixed point theory and measure theory to investigate the existence of unique solutions for integro differential equations with reflection (IDE-R). Using almost automorphic functions, we study the solutions of these equations, which are of pseudo almost automorphic ($\mathcal{PAA}$) type, by introducing the Mittag-Leffler function. Finally, we present an example we illustrate the application of the main results obtained.

1. Introduction

In the extensive research on differential equations in the literature, different unique solutions such as periodic, almost periodic, and automorphic have been obtained for these equations and generalizations and ideas are presented in different fields (also researchers considered weighted pseudo almost periodic functions which is a generalization of pseudo almost periodicity functions) [1, 2].

The main purpose in this paper is to investigate the existence of solutions for the IDE-R, which is defined as follows (considering the continuous

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*

Key words and phrases. Integro differential equation; reflection; Mittag-Leffler function.

* Speaker.

* Speaker.

functions of $k, \varphi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$):

$$\begin{aligned}\psi'(w) &= \tau\psi(w) + \kappa\psi(-w) + h(w) + k(w, \psi(\vartheta(w)), \psi(\vartheta(-w))) \\ &+ \int_{w}^{+\infty} \mathcal{L}(z-w)\varphi(z, \psi(\vartheta(w)), \psi(\vartheta(-w)))dw \\ &+ \int_{-w}^{+\infty} \mathcal{L}(z+w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz, \ w \in \mathbb{R},\end{aligned} \tag{1.1}$$

for $\tau \in \mathbb{R}, \ \kappa \in \mathbb{R}^*$.

2. Basic concepts

Definition 2.1. For Banach space $\mathcal{Y}$ and every $k \in \mathcal{C}(\mathbb{R}, \mathcal{Y})$, suppose that (z_ℓ) is a real sequence. If there is a sub-sequence (z_{ℓ_k}) such that $\lim_{\ell_k\to\infty} k(r + z_{\ell_k}) = k(r)$ and $\lim_{\ell_k\to\infty} h(r - z_{\ell_k}) = k(r)$, then, k is said to be almost automorphic or $k \in \mathcal{AA}(\mathbb{R}, \mathcal{Y})$, for every $r \in \mathbb{R}$.

Definition 2.2 ([3])**.** We consider a ζ-field $\mathcal{Z}$ as type Lebesque of $\mathbb{R}$ and suppose $\mathcal{M}$ is the space of all positive measures on $\mathcal{Z}$. Then $\eta \in \mathcal{M}$ if

(*1*) $\eta([\tau, \kappa]) < \infty$, for all $\tau \le \kappa \in \mathbb{R}$,
(*2*) $\eta(\mathbb{R}) = +\infty$.

Definition 2.3 ([4])**.** Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathcal{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is bounded continuous is called η-ergodic, $k \in \mathcal{E}(\mathbb{R}, \mathcal{Y}, \eta)$, if $\lim_{\mathtt{s}\to\infty} \frac{1}{\eta([-\mathtt{s},\mathtt{s}])} \int_{[-\mathtt{s},\mathtt{s}]} \|k(w)\| d\eta(w) = 0$, where $\eta([-\mathtt{s}, \mathtt{s}]) := \int_{-\mathtt{s}}^{\mathtt{s}} d\eta(r)$.

Definition 2.4 ([5])**.** Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathcal{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is continuous is called η-$\mathcal{PAA}$ if $k = h_1 + \mathtt{u}_1$, where h_1 is an almost automorphic function ($h_1 \in \mathcal{AA}(\mathbb{R}, \mathcal{Y})$) and $\mathtt{u}_1$ is an ergodic function.

To prove the main results of this paper, we consider the following hypotheses:

($\boldsymbol{\mathcal{N}_1}$) There exist a continuous and increasing function $\vartheta : \mathbb{R} \to \mathbb{R}$ such that for all $v \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, we have $vo\vartheta \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$.

($\boldsymbol{\mathcal{N}_2}$) For every $\gamma \in \mathbb{R}$, there exist $\vartheta > 0$ and a bounded interval J such that for positive measure η, we have $\eta(\{\tau + \gamma : \tau \in \mathcal{U}\}) \le \vartheta\eta(\mathcal{U})$, whenever $\mathcal{U} \in \mathcal{Z}$ satisfies $\mathcal{U} \cap J = \emptyset$.

($\boldsymbol{\mathcal{N}_3}$) There exist $\jmath, \ell > 0$ such that for all $\mathcal{U} \in \mathcal{Z}$,

$$\eta(-\mathcal{U}) \le \jmath + \ell\eta(\mathcal{U}).$$

($\boldsymbol{\mathcal{N}_4}$) There is a function $\rho : \mathbb{R} \to \mathbb{R}^+$ such that for all $\mathtt{E} \in \mathbb{B}(\mathbb{R})$, $\eta \in \mathcal{M}$ and $\eta_\vartheta(\mathtt{E}) = \eta\left(\vartheta^{-1}(\mathtt{E})\right)$ we have $d\eta_\vartheta(r) \le \rho(r)d\eta(r)$, ρ is also continuous, strictly increasing and

$$\limsup \frac{\eta[-\mathtt{P}(\mathtt{a}), \mathtt{P}(\mathtt{a})]}{\eta[-\mathtt{a}, \mathtt{a}]} \mathtt{Q}(\mathtt{P}(\mathtt{a})) < +\infty,$$

where $\mathtt{P}(\mathtt{a}) = |\vartheta(\mathtt{a})| + |\vartheta(-\mathtt{a})|$ and $\mathtt{Q}(\mathtt{P}(\mathtt{a})) = \sup_{r\in[-\mathtt{P}(\mathtt{a}),\mathtt{P}(\mathtt{a})]\rho(\mathtt{P})}$.

($\boldsymbol{\mathcal{N}_5}$) Given $\rho = \sqrt{\tau^2 - \kappa^2}$, where $\tau > \kappa$, the following holds

$$\mathtt{D}_1(\rho, \eta) := \sup_{\mathtt{s}>0} \left\{ \int_{-\mathtt{s}}^{\mathtt{s}} \sum_{\ell=0}^{\infty} \frac{(-\rho(r+\mathtt{s}))^{\ell}}{\Gamma(\ell\alpha+\theta)} d\eta(r) \right\} < \infty,$$

and

$$\mathtt{D}_2(\rho, \eta) := \sup_{\mathtt{s}>0} \left\{ \int_{-\mathtt{s}}^{\mathtt{s}} \sum_{\ell=0}^{\infty} \frac{(-\rho(-r+\mathtt{s}))^{\ell}}{\Gamma(\ell\alpha+\theta)} d\eta(r) \right\} < \infty.$$

($\boldsymbol{\mathcal{N}_6}$) $k : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz coefficient $\mathrm{H}_k > 0$ such that

$$|k(r, v_1, w_1) - k(r, v_2, w_2)| \leq \mathrm{H}_k \left(|v_1 - v_2| + |w_1 - w_2| \right),$$

for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_7}$) $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz coefficient $\mathrm{H}_\varphi > 0$ such that

$$|\varphi(r, \psi_1, \psi_2) - \psi(r, F_1, F_2)| < \mathrm{H}_\varphi \left(|\psi_1 - F_1| + |\psi_2 - F_2| \right),$$

for all $\psi_1, \psi_2, F_1, F_2 \in \mathbb{R}$.

($\boldsymbol{\mathcal{N}_8}$) There exists $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g = \int_0^{+\infty} \mathcal{L}(w) dw < \infty$.

To prove the results we consider two states for the Lipschitz coefficients of the functions above. In one state (above) these coefficients are constant and in the second state (below) they are not constant. In the following, we express the necessary conditions according to the second state.

($\boldsymbol{\mathcal{N}_9}$) $k : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz function $\mathrm{H}_k \in \mathrm{L}^p(\mathbb{R}, \mathbb{R}, dv) \cap \mathrm{L}^p(\mathbb{R}, \mathbb{R}, d\eta)$ such that

$$|k(r, v_1, w_1) - k(r, v_2, w_2)| \leq \mathrm{H}_k(r) \left(|v_1 - v_2| + |w_1 - w_2| \right),$$

where $\eta \in \mathcal{M}$, $p > 1$ and for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_{10}}$) $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz function $\mathrm{H}_\varphi \in \mathrm{L}^p(\mathbb{R}, \mathbb{R}, dv) \cap \mathrm{L}^p(\mathbb{R}, \mathbb{R}, d\eta)$ such that

$$|\varphi(r, v_1, w_1) - \varphi(r, v_2, w_2)| \leq \mathrm{H}_\varphi(r) \left(|v_1 - v_2| + |w_1 - w_2| \right),$$

where $\eta \in \mathcal{M}$, $p > 1$ and for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_{11}}$) There exists $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\int_0^{+\infty} (\mathcal{L}(w))^{\gamma} dw < +\infty, \text{ for all } \gamma > 1.$$

3. Existence of a unique η-$\mathcal{PAA}$ solution for equation (1.1) in two-states

Theorem 3.1. *Let $k, \varphi \in \mathcal{PAP}(\mathbb{R}, \mathbb{R}, \eta)$ and assume that ($\boldsymbol{\mathcal{N}_4}$)-($\boldsymbol{\mathcal{N}_8}$), ($\boldsymbol{\mathcal{N}_1}$)-($\boldsymbol{\mathcal{N}_3}$) are satisfied. Then equation (1.1) has a unique η-$\mathcal{PAA}$ solution if*

$$\frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\rho \left(\sum_{\ell=0}^{\infty} \frac{(\rho r)^{\ell}}{\Gamma(\ell\alpha+\theta)} \sum_{\ell=0}^{\infty} \frac{(-\rho)^{\ell} r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)} \right)} (\mathrm{H}_k + 2g\mathrm{H}_\varphi) < 1.$$

Theorem 3.2. *Consider $k,\varphi \in \mathcal{PAP}(\mathbb{R}\times\mathbb{R}^2,\mathbb{R},\eta)$ and assume that conditions $(\boldsymbol{\mathcal{N}_1})$-$(\boldsymbol{\mathcal{N}_5})$ and $(\boldsymbol{\mathcal{N}_9})$-$(\boldsymbol{\mathcal{N}_{10}})$ are satisfied. Then, equation (1.1) has a unique η-$\mathcal{PAA}$ solution if*

$$\|\mathrm{H}_k\|_{L^p(\mathbb{R},\mathbb{R},dv)} + 2\bigg(\int_0^{+\infty}(\mathcal{L}(w))^q\bigg)^{\frac{1}{q}}\|\mathrm{H}_\varphi\|_{L^p(\mathbb{R},\mathbb{R},dv)} < \frac{\sum_{\ell=0}^{\infty}\frac{((-\rho)^{\frac{1}{q}}r)^\ell}{\rho q^{\frac{1}{q}}\Gamma(\ell\alpha+\theta)}\sum_{\ell=0}^{\infty}\frac{(-\rho)^{\frac{\ell}{q}}r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}}{|\rho-\tau|+|\rho+\tau|+2|\kappa|}, \tag{3.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Example 3.3. If we consider an equation of type (1.1) such that $\mathcal{L}(z) = \sum_{\ell=0}^{\infty}\frac{(-|z|)^\ell}{\Gamma(\alpha\ell+\theta)}$ for all $z\in\mathbb{R}^+$. In order for condition $(\boldsymbol{\mathcal{N}_1})$ to be true, we set $\vartheta(r) = \frac{1}{r+1} - e$. By putting $\tau = 4, \kappa = \sqrt{7}, \rho = \sqrt{\tau^2-\kappa^2} = 3, k(r,v,w) = \varphi(r,v,w) = \frac{1}{15}\sum_{\ell=0}^{\infty}\frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)}[\sin v + \cos w]$, $p = q = \frac{1}{2}, \alpha = 1$, $\theta = 1$ and $\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \frac{1}{15}$ and $\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} \leq \frac{1}{15}\sqrt{\exp(1)}$ condition $(\boldsymbol{\mathcal{N}_7})$ is satisfiedn where $\mathrm{H}_k(r) = \mathrm{H}_\varphi(r) = \frac{1}{15}\sum_{\ell=0}^{\infty}\frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)}$. Therefore

$$\begin{aligned}&\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} + 2\left(\int_0^{+\infty}(\mathcal{L}(w))^2dw\right)^{\frac{1}{2}}\|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)}\\ &= \frac{\sqrt{2}+1}{15} < \frac{e^{18}}{|\rho-\tau|+|\rho+\tau|+2|\kappa|} = \frac{e^{18}}{8+2\sqrt{7}}.\end{aligned}$$

all the conditions of theorem 3.2 are satisfied and equation has a unique η-$\mathcal{PAP}$ solution.

References

1. Bochner, S. *Continuous mappings of almost automorphic and almost periodic functions.* Proc. Nat. Acad. Sci. U.S.A. 52 (1964), 907–910.
2. Ait Dads, El Hadi; E, Khalil; Miraoui, M. *(μ,ν)-pseudo almost automorphic solutions for some non-autonomous differential equations.* Internat. J. Math. 26 (2015), no. 11, 1550090, 21 pp.
3. Blot, J; Cieutat, Ph; Ezzinbi, Kh. *Measure theory and pseudo almost automorphic functions: new developments and applications.* Nonlinear Anal. 75 (2012), no. 4, 2426–2447.
4. Diagana, T; Ezzinbi, Kh; Miraoui, M. *Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory.* Cubo 16 (2014), no. 2, 1–31. MR3237503
5. Dads, Elhadi A; Fatajou, S; Khachimi, L. *Pseudo almost automorphic solutions for differential equations involving reflection of the argument.* ISRN Math. Anal. 2012, Art. ID 626490, 20 pp.