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# STABILITY RESULTS OF FRACTIONAL DIFFERENTIAL EQUATIONS IN THE HILFER SENSE IN MATRIX-VALUED MENGER SPACES

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Abstract. In the present paper, we use some special functions to present the notion of multi-stability and obtain an approximation of fractional differential equations through a fixed point theory. Moreover, some UH stability results for the governing models in different cases are gained.

## 1. Introduction

Assume the non homogenous vector-valued fractional differential equation given by

$$ {}^{\mathcal{H}}\mathbf{D}^{a,\sigma}\Phi(\lambda) = \theta\Phi(\lambda) + \Phi(\lambda)\overline{\theta} + \Psi(\lambda), \qquad \Phi(0) = \lambda_0, \tag{1.1} $$

in which ${}^{\mathcal{H}}\mathbf{D}^{a,\sigma}$ is the Hilfer fractional derivative of order $a$ and parameter $\sigma$, and $0 < \lambda < \omega < +\infty$. Assume $\zeta_n$ be a matrix of $n^2$.

Consider the following cases:

(1) : $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_1$,
(2) : $\overline{\theta} = 0_{1\times 1}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times 1}$,
(3) : $\overline{\theta} = 0_{m\times m}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times m}$,

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(4) : $\overline{\theta} \in \zeta_m$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times m}$.

In case (1), we use some special functions to study a class of matrix-valued random controllers and also to present the notion of multi-stability. Next, we show the equation (1.1) is the multi-stable. In other cases, via the fixed point theory, we study the UH stability for the equation (1.1).

## 2. Preliminaries

Assume $\mathcal{O} = [0,1]$ and

$$\text{diag}\zeta_n(\mathcal{O}) = \left\{ \begin{bmatrix} \upsilon_1 & & \\ & \ddots & \\ & & \upsilon_n \end{bmatrix} = \text{diag}[\upsilon_1, \cdots, \upsilon_n],\ \upsilon_1, ..., \upsilon_n \in \mathcal{O} \right\}.$$

We denote $\upsilon := \text{diag}[\upsilon_1, \cdots, \upsilon_n] \preceq \beta := \text{diag}[\beta_1, \cdots, \beta_n]$ if $\upsilon_i \leq \beta_i$ for all $1 \leq i \leq n$.

Next, we define generalized t-norm (GTN) on $\text{diag}\zeta_n(\mathcal{O})$.

**Definition 2.1.** A GTN on $\text{diag}\zeta_n(\mathcal{O})$ is an operation $\circledast : \text{diag}\zeta_n(\mathcal{O}) \times \text{diag}\zeta_n(\mathcal{O}) \to \text{diag}\zeta_n(\mathcal{O})$ satisfying the conditions below:

(1) $(\forall \upsilon \in \text{diag}\zeta_n(\mathcal{O}))(\upsilon \circledast \mathbf{1}) = \upsilon)$ (boundary condition);

(2) $(\forall(\upsilon, \beta) \in (\text{diag}\zeta_n(\mathcal{O}))^2)(\upsilon \circledast \beta = \beta \circledast \upsilon)$ (commutativity);

(3) $(\forall(\upsilon, \beta, \gamma) \in (\text{diag}\zeta_n(\mathcal{O})^3)(\upsilon \circledast (\beta \circledast \gamma) = (\upsilon \circledast \beta) \circledast \gamma)$ (associativity);

(4) $(\forall(\upsilon, \upsilon', \beta, \beta') \in (\text{diag}\zeta_n(\mathcal{O}^4)(\upsilon \preceq \upsilon'$ and $\beta \preceq \beta' \Longrightarrow \upsilon \circledast \beta \preceq \upsilon' \circledast \beta'$ (monotonicity).

For any $\upsilon, \beta \in \text{diag}\zeta_n(\mathcal{O})$ and any sequences $\{\upsilon_k\}$ and $\{\beta_k\}$ converging to $\upsilon$ and $\beta$, if we get $\lim_k(\upsilon_k \circledast \beta_k) = \upsilon \circledast \beta$, thus $\circledast$ on $\text{diag}\zeta_n(\mathcal{O})$ is continuous.

Presume $\mathcal{Z}^+$, the set of matrix distribution functions, including increasing and left continuous maps $\psi : \mathbb{R} \cup \{-\infty, \infty\} \to \text{diag}\zeta_n(\mathcal{O})$ s.t. $\psi_0 = \mathbf{0}$ and $\psi_{+\infty} = \mathbf{1}$. Now $\Delta^+ \subseteq \mathcal{Z}^+$ are all mappings $\psi \in \mathcal{Z}^+$ for which $\ell^-\psi_\varepsilon = \lim_{\sigma\to\varepsilon^-} \psi_\sigma = \mathbf{1}$.

In $\mathcal{Z}^+$, we define " $\preceq$ " as: $\Psi \preceq \psi \iff \Psi_\varepsilon \preceq \psi_\varepsilon,\ \forall \varepsilon \in \mathbb{R}$. In addition

$$\nabla_r^j = \begin{cases} \mathbf{0}, & \text{if } r \leq j, \\ \mathbf{1}, & \text{if } r > j \end{cases}$$

belongs to $\mathcal{Z}^+$ and for each matrix distribution function $\psi$, $\psi \preceq \nabla^0$.

**Definition 2.2.** Assume $\circledast$ be a continuous GTN, $\mathcal{J}$ be a linear space, and $\psi : \mathcal{J} \to \Delta^+$ be a matrix distribution function. The triple $(\mathcal{J}, \psi, \circledast)$ is called a matrix Menger normed space if we get

(1) $\psi_\varepsilon^j = \nabla_\varepsilon^0$ for all $\varepsilon > 0$ if and only if $j = 0$;

(2) $\psi_\varepsilon^{\nu j} = \psi^j_{\frac{\varepsilon}{|\nu|}}$ for any $s \in \mathcal{J}$ and $\nu \in \mathbb{C}$ with $\nu \neq 0$;

(3) $\psi_{\varepsilon+\varepsilon'}^{j+j'} \succeq \psi_\varepsilon^j \circledast \psi_{\varepsilon'}^{j'}$ for any $j, j' \in \mathcal{J}$ and $\varepsilon, \varepsilon' \geq 0$.

A complete matrix Menger normed space is called a matrix Menger Banach space.

For more details, we refer to [1, 2, 3].

## 3. Multi-stability for (1.1), when $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_{1,1}$,

Assume the following random controller given by

$$\begin{aligned}\Lambda\Big(-\frac{|\lambda|^a}{\Theta\varepsilon}\Big) = \mathrm{diag}\Big[{}_0\mathbb{H}_0\Big(-\frac{|\lambda|^a}{\Theta\varepsilon}\Big), {}_0\mathbb{H}_1[e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], {}_2\mathbb{H}_1[d_1, d_2; e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], \quad (3.1)\\ {}_1\mathbb{H}_1[d_1; e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], {}_s\mathbb{H}_r\Big[-\frac{|\lambda|^a}{\Theta\varepsilon}\Big|^{(d_1,D_1),\ldots,(d_s,D_s)}_{(e_1,E_1),\ldots,(e_r,E_r)}\Big], {}^v_s\mathbb{H}^w_r\Big[-\frac{|\lambda|^a}{\Theta\varepsilon}\Big|^{(d_j,D_j)_{1,s}}_{(e_j,E_j)_{1,r}}\Big]\Big]\end{aligned}$$

where $\Theta > 0$, $\varepsilon \in (0, \infty)$, $0 < a < 1$, and ${}_0\mathbb{H}_0, {}_0\mathbb{H}_1, {}_1\mathbb{H}_1, {}_s\mathbb{H}_r, {}^v_s\mathbb{H}^w_r$ are Exponential function, Mittag–Leffler function, Hypergeometric function, Wright function, Fox–Wright function, and Fox's H–function respectively. for more details see [4].

Notice that the Fox's $\mathbb{H}$-function is defined by

$${}^v_s\mathbb{H}^w_r\Big[X\Big|^{(d_j,D_j)_{1,s}}_{(e_j,E_j)_{1,r}}\Big] := \frac{1}{2\pi i}\int_Z O(Y)X^Y dY, \quad (3.2)$$

where $i^2 = -1$, $X \in \mathbb{C}\backslash\{0\}$, $X^Y = \exp(Y[\log|X| + i\arg(X)])$, $\log|X|$ denotes the natural logarithm of $|X|$ and $\arg(X)$ is not necessarily the principal value. For convenience,

$$O(Y) := \frac{\prod_{j=1}^{v}\Gamma(e_j - E_jY)\prod_{j=1}^{w}\Gamma(1 - d_j + D_jY)}{\prod_{j=v+1}^{r}\Gamma(1 - e_j + E_jY)\prod_{j=w+1}^{s}\Gamma(d_j - D_jY)},$$

where an empty product is interpreted as 1, and the integers $v, w, s, r$ satisfy the inequalities $0 \le w \le s$ and $1 \le v \le r$. Assume the coefficients

$$D_j > 0 \ (j = 1, \ldots, s) \quad \text{and} \quad E_j > 0 \ (j = 1, \ldots, r),$$

and the complex parameters

$$d_j \ (j = 1, \ldots, s) \quad \text{and} \quad e_j \ (j = 1, \ldots, r)$$

are constrained such that no poles of integrand in (3.2) coincide, and $Z$ is a suitable contour of the Mellin-Barnes type (in the complex $Y$-plane) which separates the poles of one product from the others. Further, if we assume

$$\ell := \sum_{j=1}^{w} D_j - \sum_{j=w+1}^{s} D_j + \sum_{j=1}^{v} E_j - \sum_{j=\mathsf{Q}+1}^{r} E_j > 0,$$

then the integral in (3.2) converges absolutely and defines the $\mathbb{H}$-function, which is analytic in the sector: $|\arg(X)| < \frac{1}{2}\ell\pi$ and with the point $X = 0$ being tacitly excluded. Actually, the $\mathbb{H}$–function makes sense and also defines an analytic function of $X$ when either

$$E := \sum_{j=1}^{s} D_j - \sum_{j=1}^{r} E_j < 0 \quad \text{and} \quad 0 < |X| < \infty,$$

or

$$E = 0 \quad \text{and} \quad 0 < |X| < R := \prod_{j=1}^{s} D_j^{-D_j} \prod_{j=1}^{r} E_j^{E_j}.$$

**Definition 3.1.** Equation (1.1) has Multi-stability, with respect to $\Lambda(-\frac{|\lambda|^a}{\Theta\varepsilon})$, if there is $\hbar > 0$, s.t for all $\Theta > 0$, and all solution $\Phi$ to (3.3), there is a solution $\Phi'$ to (1.1), with $\psi_\varepsilon^{\Phi-\Phi'} \succeq \Lambda\bigg(-\dfrac{|\lambda|^a}{\hbar\Theta\varepsilon}\bigg)$, where $\varepsilon \in (0,\infty)$.

**Theorem 3.2.** *Let (1.1), when* $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_{1,1}$, *also Let*

$$\psi_\varepsilon^{\mathcal{H}\mathbf{D}^{\sigma,\delta}\Phi(\lambda)-\Psi(\lambda)} \succeq \Lambda\bigg(-\frac{|\lambda|^a}{\Theta\varepsilon}\bigg). \tag{3.3}$$

*Then (1.1) is Multi-stable, with respect to* $\Lambda(-\frac{|\lambda|^a}{\Theta\varepsilon})$.

## 4. UH stability for (1.1), when $\overline{\theta} = 0_{1\times 1}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,1}$,

**Theorem 4.1.** *If each the eigenvalues of* $\theta$ *satisfy* $|\arg(\mu(\theta))| > \dfrac{a\pi}{2}$. *Then, (1.1) is UH stable.*

## 5. UH stability for (1.1), when $\overline{\theta} = 0_{m\times m}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,m}$

**Theorem 5.1.** *If any the eigenvalues of* $\theta$ *satisfy* $|\arg(\mu(\theta))| > \dfrac{a\pi}{2}$. *Then, (1.1) is UH stable.*

## 6. UH stability for (1.1), when $\overline{\theta} \in \zeta_m$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,m}$

**Theorem 6.1.** *Assume all the eigenvalues of* $\theta$ *and* $\overline{\theta}$ *satisfy*

$$|\arg(\mu(\theta))| > \frac{a\pi}{2}, \quad \pi \geq |\arg(\mu(\overline{\theta}))| \geq k \quad (\frac{a\pi}{2} < k < \min\{\pi, \pi a\}).$$

*then (1.1) is UH stable.*

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