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ON GENERALIZED FEJÉR INEQUALITY AND A CLASS OF FRACTIONAL INTEGRALS

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Abstract. A real mapping $\mathcal{M}_{\mathcal{F}}^{\omega}(t)$ is introduced, a generalized form of Fejér's inequality is obtained and some new and generalized inequalities in connection with fractional integrals and monotone functions are given.

1. Introduction and Preliminaries

Lipót Fejér (1880-1959) in 1906 [4], while studying trigonometric polynomials, discovered the following integral inequalities which later became known as Fejér's inequality (in some references is separated to the left and right):

$$\mathcal{F}\Big(\frac{a+b}{2}\Big)\int_a^b \mathcal{G}(x)dx \leq \int_a^b \mathcal{F}(x)\mathcal{G}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\int_a^b \mathcal{G}(x)dx, \quad (1.1)$$

where $\mathcal{F}$ is a convex function ([9]) in the interval (a,b) and $\mathcal{G}$ is a positive function in the same interval such that

$$\mathcal{G}(a+t)=\mathcal{G}(b-t), \qquad 0 \leq t \leq \frac{a+b}{2},$$

i.e., $y=\mathcal{G}(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{a+b}{2},0)$ and is normal to the x-axis. In fact the Fejér's

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inequality (1.1), is the weighted version of celebrated Hermite-Hadamard's inequality for convex function $f : [a, b] \to \mathbb{R}$:

$$\mathcal{F}\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{1.2}$$

Our aim in this paper is obtaining a generalized form of Fejér's inequality and applying it to give some new and generalized inequalities in connection with fractional integrals and monotone functions. We introduce a real mapping $\mathcal{M}_f^{\omega}(t)$ and obtain some basic properties for it. Also we use the concept of h-convexity introduced by S. Varošanec in 2006 ([13]):

Definition 1.1. We say that a non-negative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex or $f \in SX(h, I)$, if for non-negative function $h : (0, 1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ $(h \not\equiv 0)$, all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$\mathcal{F}(\alpha x + (1-\alpha)y) \leq h(\alpha)\mathcal{F}(x) + h(1-\alpha)\mathcal{F}(y).$$

f is said to be h-concave or $f \in SV(h, I)$, If above inequality is reversed.

The mapping $\mathcal{M}_f^{\omega}(t)$. For two real numbers $a < b$, consider integrable functions $f : [a, b] \to \mathbb{R}$ and $\omega : [a, b] \to \mathbb{R}^+ \cup \{0\}$. Define a mapping $\mathcal{M}_f^{\omega}(t) : [0, 1] \to \mathbb{R}$ as

$$\mathcal{M}_f^{\omega}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)\omega(x)dx,$$

such that

$$m_t(\mathcal{L}, \mathcal{R}) = \min\{\mathcal{L}(t), \mathcal{R}(t)\}, M_t(\mathcal{L}, \mathcal{R}) = \max\{\mathcal{L}(t), \mathcal{R}(t)\}$$

where $\mathcal{L}(t) : [0, 1] \to [a, b]$ and $\mathcal{R}(t) : [0, 1] \to [a, b]$ are considered as the following:

$$\mathcal{L}(t) = tb + (1-t)a, \mathcal{R}(t) = ta + (1-t)b$$

for any $t \in [0, 1]$. Note that

$$\mathcal{M}_f^{1}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)dx,$$

where by 1, we mean $\omega \equiv 1$.

Some basic properties for the mapping $\mathcal{M}_f^{\omega}(t)$ are obtained in the following:

Proposition 1.2. *Consider two functions $f : [a, b] \to \mathbb{R}$ and $\omega : [a, b] \to \mathbb{R}^+ \cup \{0\}$. Then*
(i) For all $t \in [0, 1]$,

$$\mathcal{M}_f^{\omega}(t) = \mathcal{M}_f^{\omega}(1-t),$$

which shows $\mathcal{M}_f^{\omega}(t)$ is symmetric on $[a, b]$ with respect to $\frac{1}{2}$.

(ii) For symmetric ω on $[a,b]$ with respect to $\frac{a+b}{2}$ and $p,q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\mathcal{M}_f^{\omega}(t)\right|\leq \|f\|_p\|\omega\|_q.$$

Also if $m_t(\mathcal{L},\mathcal{R})=\mathcal{L}(t)$, then

$$\left|\mathcal{M}_f^{\omega}(t)\right|\leq (\frac{1}{2})^{\frac{1}{q}}[t(b-a)]^{\frac{1}{p}}\|\omega\|_q\|f\|_\infty,$$

and if $m_t(\mathcal{L},\mathcal{R})=\mathcal{R}(t)$, then

$$\left|\mathcal{M}_f^{\omega}(t)\right|\leq (\frac{1}{2})^{\frac{1}{q}}[(1-t)(b-a)]^{\frac{1}{p}}\|\omega\|_q\|f\|_\infty.$$

(iii) Suppose that the function $(f\omega)(x)=f(x)\omega(x)$ is convex on $[a,b]$. If $m_t(\mathcal{L},\mathcal{R})=\mathcal{R}(t)$ for some $t\in[0,1)$, then the function $\frac{\mathcal{M}_f^{\omega}(t)}{1-t}$ is convex. Also if $m_t(\mathcal{L},\mathcal{R})=\mathcal{L}(t)$ for some $t\in(0,1]$, then the function $\frac{\mathcal{M}_f^{\omega}(t)}{t}$ is convex.
(iv) Suppose that f and ω are two continuous functions on $[a,b]$. If f is nonnegative (nonpositive) on $[a,b]$, then the function $\mathcal{M}_f^{\omega}(t)$ is increasing (decreasing) on $[0,\frac{1}{2})$ and it is decreasing (increasing) on $(\frac{1}{2},1]$. Also $\mathcal{M}_f^{\omega}(t)$ has a relative extreme point in $t=\frac{1}{2}$. If $\omega\not\equiv 0$, then corresponding to any $x\in[a,b]\setminus\{\frac{a+b}{2}\}$ satisfying

$$f(x)+f(a+b-x)=0,$$

there exists a critical point for $\mathcal{M}_f^{\omega}(t)$.

2. Generalization and refinement of Fejér's Inequality

The following result presents a new and generalized type of the celebrated Fejér's inequality in connection with h-convex functions.

Theorem 2.1. *Consider two integrable functions $f:[a,b]\to\mathbb{R}$ and $w:[a,b]\to\mathbb{R}^+\cup\{0\}$ such that f is h-convex and ω is symmetric with respect to $\frac{a+b}{2}$. For all $t\in[0,1]$, the following inequality hold:*

$$\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)dx\leq\int_a^b f(x)\omega(x)dx-\mathcal{M}_f^{\omega}(t) \tag{2.1}$$
$$\leq\frac{|\mathcal{R}(t)-\mathcal{L}(t)|[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{(\mathcal{L}(t)-\mathcal{R}(t))}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)}h\Big(\frac{x-\mathcal{R}(t)}{\mathcal{L}(t)-\mathcal{R}(t)}\Big)\omega(x)dx$$
$$=\frac{|\mathcal{R}(t)-\mathcal{L}(t)|\Big([f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)]\Big)}{(\mathcal{R}(t)-\mathcal{L}(t))}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)}h\Big(\frac{x-\mathcal{L}(t)}{\mathcal{R}(t)-\mathcal{L}(t)}\Big)\omega(x)dx.$$

Inequality (2.1) is a generalization of many Fejér's type inequalities obtained for h-convex functions in literature. However if we set $h(s)=s$ in

(2.1), then the following inequality holds:

$$f\Big(\frac{a+b}{2}\Big)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)dx \le \int_a^b f(x)\omega(x)dx - \mathcal{M}_f^{\omega}(t) \tag{2.2}$$
$$\le \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{R}(t)-\mathcal{L}(t)|}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)}(x-\mathcal{L}(t))\omega(x)dx$$
$$= \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{L}(t)-\mathcal{R}(t)|}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)}(x-\mathcal{R}(t))\omega(x)dx,$$

Inequality (2.2) is a new generalized Fejér's type inequality related to the convex functions.

If we set $t=0,1$ in (2.1) ($\mathcal{M}_f^{\omega}(0)=\mathcal{M}_f^{\omega}(1)=0$), then we recapture the following Fejér's type inequality related to the h-convex functions obtained in [2]:

$$\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b\omega(x)dx \le \int_a^b f(x)\omega(x)dx \tag{2.3}$$
$$\le (b-a)[f(a)+f(b)]\int_0^1 h(s)\omega\big(sa+(1-s)b\big)ds,$$

Also we can obtain a new h-convex version of Fejér's inequality:

$$\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b\omega(x)dx \le \int_a^b f(x)\omega(x)dx \tag{2.4}$$
$$\le \frac{f(a)+f(b)}{2}\int_a^b H\Big(\frac{b-x}{b-a}\Big)\omega(x)dx$$
$$= \frac{f(a)+f(b)}{2}\int_a^b H\Big(\frac{x-a}{b-a}\Big)\omega(x)dx.$$

If in (2.3) and (2.4) we consider $\omega\equiv 1$, then we recapture the following result obtained in [11]:

$$\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\le\frac{1}{b-a}\int_a^b f(x)dx\le\big[f(a)+f(b)\big]\int_0^1 h(s)ds,$$

which is the Hermite-Hadamard's inequality related to h-convex functions.

3. Fractional Integrals

In this section, we introduce a new class of fractional integrals and just consider one special case which is known in literature as Riemann-Liouville fractional integrals (see [5, 7, 8, 10]) to find some hermite-hadamard's type inequalities for it by using generalized Fejér inequality obtained in previous section.

For $t \in [0,1]\backslash\{\frac{1}{2}\}$ consider a bifunction $G : [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]\times[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})] \to \mathbb{R}^+ \cup \{0\}$ and define the following class of fractional integrals:

$$\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x) = \int_{m_t(\mathcal{L},\mathcal{R})}^{x} G(x,u)f(u)du, \qquad x > m_t(\mathcal{L},\mathcal{R})$$

and

$$\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x) = \int_{x}^{M_t(\mathcal{L},\mathcal{R})} G(x,u)f(u)du, \qquad x < M_t(\mathcal{L},\mathcal{R})$$

if above integrals exist.
Now we discuss a special case of $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ and obtain some results in connection with Theorem 2.1.

In $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ for $\alpha > 0$, consider

$$G(x,u) = \frac{1}{\Gamma(\alpha)}|x-u|^{\alpha-1}, \; x,u \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})].$$

So we achieve the following generalized Riemann-Liouville fractional integrals:

$$\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_{m_t(\mathcal{L},\mathcal{R})}^{x} (x-u)^{\alpha-1}f(u)du \qquad x > m_t(\mathcal{L},\mathcal{R})$$

and

$$\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{M_t(\mathcal{L},\mathcal{R})} (u-x)^{\alpha-1}f(u)dt \qquad M_t(\mathcal{L},\mathcal{R}) < x.$$

Fractional integrals $\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x)$ and $\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x)$ in special case ($t = 0,1$) reduce to $J^{\alpha}_{a^+}f(x)$ and $J^{\alpha}_{b^-}f(x)$ respectively, which are known as Riemann-Liouville fractional integrals. Now in Theorem 2.1, consider

$$\omega(x) = \frac{(M_t(\mathcal{L},\mathcal{R})-x)^{\alpha-1} + (x-m_t(\mathcal{L},\mathcal{R}))^{\alpha-1}}{\Gamma(\alpha)}, \quad x \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})].$$

It is not hard to see that w is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$ and also nonnegative. Also the following results hold:

$$\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx = \frac{2\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{\alpha}}{\Gamma(\alpha+1)} = \frac{2(b-a)^{\alpha}|1-2t|^{\alpha}}{\Gamma(\alpha+1)},$$

$$\int_a^b f(x)\omega(x)dx - \mathcal{M}^{\omega}_f(t) = \mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R})),$$

and

$$\int_0^1 h(s)[\omega \circ A_s](m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R}))ds = (M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{\alpha-1}\int_0^1 H(s)s^{\alpha-1}ds.$$

Above results altogether imply that:

$$\frac{1}{h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}|1-2t|^{\alpha}}[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))] \tag{3.1}$$

$$\leq \alpha[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)]\int_0^1 H(s)s^{\alpha-1}ds,$$

for $t\in[0,1]\setminus\{\frac{1}{2}\}$.

In the case that $h(s)=s$, from (3.1) we reach the following inequality which is generalization of inequality (2.1) obtained in [12]:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}|1-2t|^{\alpha}}[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))]$$
$$\leq \frac{f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)}{2}.$$

Also for $\alpha=1$, we obtain a generalization of inequality (2.1) presented in [11]:

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)|1-2t|}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)dx \leq [f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)]\int_0^1 h(s)ds.$$

4. Refinements for Hermite-Hadamard's Inequality by Monotone Functions

In this section, we obtain some refinements for Hermite-Hadamard's inequality by the use of fractional integrals discussed in previous section provided that considered functions are nonnegative and monotone. We focus on Riemann-Liouville fractional integrals but results can be extended to many classes of fractional integrals. We need the following result which is a consequence of Theorem 1 in [1](see also [3, 6]).

Theorem 4.1. *If f_1 and f_2 are nonnegative increasing functions on $[0,1]$, Then*

$$\int_0^1 f_1(x)dx\int_0^1 f_2(x)dx \leq \int_0^1 f_1(x)f_2(x)dx.$$

Here we give some refinements for Hermite-Hadamard's inequality by the use of fractional integrals for h-convex functions:

Theorem 4.2. *Suppose that $f:[a,b]\to\mathbb{R}$ is an integrable h-convex function and $t\in[0,1]\setminus\{\frac{1}{2}\}$. Then*
(i) For $\alpha\geq 1$, the following inequality holds if f is nonnegative and increasing

:

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{|1-2t|(b-a)} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)du \tag{4.1}$$
$$\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}} \Big[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))\Big]$$
$$\leq \alpha \Big[\frac{f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)}{2}\Big] \int_0^1 H(s)s^{\alpha-1}ds.$$

(ii) For any $\alpha > 0$ we have

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \tag{4.2}$$
$$\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}} \Big[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))\Big]$$
$$\leq \frac{\alpha}{|1-2t|(b-a)} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)du.$$

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