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APPLICATIONS OF FEJÉR'S INEQUALITY IN CONNECTION WITH EULER'S BETA AND GAMMA FUNCTIONS

MOHSEN ROSTAMIAN DELAVAR*

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
m.rostamian@ub.ac.ir

Abstract. Some new and generalized results related to the Euler's beta and gamma functions are presented by the use of generalized Fejér's Inequality.

1. Introduction and Preliminaries

Lipót Fejér (1880-1959) in 1906 [4], while studying trigonometric polynomials, discovered the following integral inequalities which later became known as Fejér's inequality (in some references is separated to the left and right):

$$\mathcal{F}\Big(\frac{a+b}{2}\Big)\int_a^b \mathcal{G}(x)dx \leq \int_a^b \mathcal{F}(x)\mathcal{G}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\int_a^b \mathcal{G}(x)dx, \quad (1.1)$$

where $\mathcal{F}$ is a convex function ([6]) in the interval (a,b) and $\mathcal{G}$ is a positive function in the same interval such that

$$\mathcal{G}(a+t)=\mathcal{G}(b-t), \qquad 0 \leq t \leq \frac{a+b}{2},$$

i.e., $y=\mathcal{G}(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{a+b}{2},0)$ and is normal to the x-axis. In fact the Fejér's

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* Speaker.

inequality (1.1), is the weighted version of celebrated Hermite-Hadamard's inequality for convex function $f : [a, b] \to \mathbb{R}$:

$$\mathcal{F}\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{1.2}$$

In this paper some new and generalized results related to the Euler's beta and gamma functions are presented by the use of generalized Fejér's Inequality. Also we use the concept of h-convexity introduced by S. Varošanec in 2006 ([9]):

Definition 1.1. We say that a non-negative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex or $f \in SX(h, I)$, if for non-negative function $h : (0, 1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ $(h \not\equiv 0)$, all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$\mathcal{F}(\alpha x + (1-\alpha)y) \leq h(\alpha)\mathcal{F}(x) + h(1-\alpha)\mathcal{F}(y).$$

f is said to be h-concave or $f \in SV(h, I)$, If above inequality is reversed.

The following result presents a new and generalized type of the celebrated Fejér's inequality in connection with h-convex functions.

Theorem 1.2. *Consider two integrable functions $f : [a, b] \to \mathbb{R}$ and $w : [a, b] \to \mathbb{R}^+ \cup \{0\}$ such that f is h-convex and ω is symmetric with respect to $\frac{a+b}{2}$. For all $t \in [0, 1]$, the following inequality hold:*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx \leq \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx \qquad (1.3)\\
&\leq \frac{|\mathcal{R}(t)-\mathcal{L}(t)|[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{(\mathcal{L}(t)-\mathcal{R}(t))}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)} h\Big(\frac{x-\mathcal{R}(t)}{\mathcal{L}(t)-\mathcal{R}(t)}\Big)\omega(x)dx\\
&= \frac{|\mathcal{R}(t)-\mathcal{L}(t)|\Big([f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)]\Big)}{(\mathcal{R}(t)-\mathcal{L}(t))}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)} h\Big(\frac{x-\mathcal{L}(t)}{\mathcal{R}(t)-\mathcal{L}(t)}\Big)\omega(x)dx,
\end{aligned}$$

where

$$m_t(\mathcal{L},\mathcal{R}) = \min\{\mathcal{L}(t),\mathcal{R}(t)\}, M_t(\mathcal{L},\mathcal{R}) = \max\{\mathcal{L}(t),\mathcal{R}(t)\}$$

and $\mathcal{L}(t) : [0, 1] \to [a, b]$, $\mathcal{R}(t) : [0, 1] \to [a, b]$ are considered as the following:

$$\mathcal{L}(t) = tb + (1-t)a, \mathcal{R}(t) = ta + (1-t)b$$

for any $t \in [0, 1]$.

Inequality (1.3) is a generalization of many Fejér's type inequalities obtained for h-convex functions in literature. However if we set $h(s) = s$ in

(1.3), then the following inequality holds:

$$f\Big(\frac{a+b}{2}\Big)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)dx \le \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx \tag{1.4}$$
$$\le \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{R}(t)-\mathcal{L}(t)|}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)}(x-\mathcal{L}(t))\omega(x)dx$$
$$= \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{L}(t)-\mathcal{R}(t)|}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)}(x-\mathcal{R}(t))\omega(x)dx,$$

2. Gamma and Beta Function

In this section, we present some inequalities and results related to gamma and beta functions. Specially by considering appropriate functions in Theorem 1.2 along with some calculations, we give a simple proof for well known Stirling's formula as well.
The Euler's integral of the second kind i.e. Gamma function [2] is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad Re(x) > 0.$$

Consider the function $f(x) = \ln\Gamma(x), x \in (0,+\infty)$ which is convex ($\Gamma(x)$ is log-convex). To see this (see also [1]), we should have

$$(\ln\Gamma)''(x) = \frac{\Gamma''(x)\Gamma(x) - (\Gamma'(x))^2}{(\Gamma(x))^2} > 0,$$

which happens by using Cauchy-Schwarz inequality [8], for

$$< f, g >= \int_0^\infty f(t)g(t)t^{x-1}e^{-t}dt, \qquad (f(t) = \ln(t),\ g \equiv 1),$$

and the fact that

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1}e^{-t}[ln(t)]^n dt. \qquad (nth\ derivative)$$

Now in Theorem 1.2, consider $h(s) = s$, $t = 0, 1$, $b = a+1$ for $a \in (0,+\infty)$ and symmetric function $\omega : [a, a+1] \to (0,+\infty)$ with respect to $a+\frac{1}{2}$. Then we obtain the following inequality:

$$\Gamma(a+\frac{1}{2}) \le \exp\Big(\frac{1}{\mathcal{K}}\int_a^{a+1}\omega(x)\ln\Gamma(x)dx\Big) \le \sqrt{\Gamma(a)\Gamma(a+1)}, \tag{2.1}$$

where $\mathcal{K} = \int_a^{a+1}\omega(x)dx$. In special case for $\omega \equiv 1$, by the Raabe's formula [5], i.e.

$$\int_a^{a+1}\ln\Gamma(x)dx = \ln\sqrt{2\pi} + a\ln(a) - a,$$

and inequality (2.1) we have

$$\Gamma(a+\frac{1}{2}) \le \sqrt{2\pi}\big(\frac{a}{e}\big)^a \le \sqrt{\Gamma(a)\Gamma(a+1)}, \tag{2.2}$$

for any $a \in (0, +\infty)$. By applying Wendel's inequality ([10]), i.e.

$$\left(\frac{a}{a+s}\right)^{1-s} \leq \frac{\Gamma(a+s)}{a^s\Gamma(a)} \leq 1,$$

in (2.2) for $s = \frac{1}{2}$, we get to

$$\sqrt{\frac{a}{a+\frac{1}{2}}} \leq \frac{\Gamma(a+\dfrac{1}{2})}{a^{\frac{1}{2}}\Gamma(a)} \leq \frac{\sqrt{2\pi a}(\dfrac{a}{e})^r}{\Gamma(a+1)} \leq 1. \tag{2.3}$$

So two results can be extracted from inequality (2.3) by using squeeze theorem [7]. The First is

$$\lim_{a\to\infty} \frac{\Gamma(a+\frac{1}{2})}{a^{\frac{1}{2}}\Gamma(a)} = 1,$$

and the second is generalization of Stirling's formula [3],

$$\Gamma(a+1) \approx \sqrt{2\pi a}(\frac{a}{e})^a \qquad as \;\; a \to \infty.$$

For the case that $a \in \mathbb{N}$, we recapture the classic Stirling's formula:

$$a! \approx \sqrt{2\pi a}(\frac{a}{e})^a \qquad as \;\; a \to \infty.$$

The Euler's integral of the first kind is known as beta function [1]:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad Re(x) > 0, Re(y) > 0.$$

To obtain some results in connection with beta function by the use of Fejér's inequality consider

$$\begin{cases} f(x) = \big(x - m_t(\mathcal{L},\mathcal{R})\big)^r, \;\; 0 < m_t(\mathcal{L},\mathcal{R}) \leq x \leq M_t(\mathcal{L},\mathcal{R}), r \in [1,\infty); \\[2ex] \omega(x) = \dfrac{\big(M_t(\mathcal{L},\mathcal{R}) - x\big)^{P-1}\big(x - m_t(\mathcal{L},\mathcal{R})\big)^{P-1}}{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^p}, \;\; 0 < m_t(\mathcal{L},\mathcal{R}) \leq x \leq M_t(\mathcal{L},\mathcal{R}); \\[2ex] h(s) = s^k, \;\; 0 \leq k \leq 1, s > 0, \end{cases}$$

where $0 < a < b$, $p > 0$ and $t \in [0,1]\backslash\{\frac{1}{2}\}$. From example 7 in [9], we deduce that f is a h-convex function on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$. Also it is not hard to see that ω is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$. The following results hold:

$$\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx = \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1} \int_0^1 x^{p-1}(1-x)^{p-1}dx$$
$$= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1}\beta(p,p),$$

$$\begin{aligned}
&\int_a^b f(x)\omega(x)dx - \mathcal{M}_f^{\omega}(t) = \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx \\
&= \frac{\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{2p+r-1}}{\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p}} \int_0^1 x^{p-1}(1-x)^{p+r-1}dx \\
&= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p+r-1}\beta(p,p+r).
\end{aligned}$$

Also

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big) = \Big(\frac{a+b}{2} - m_t(\mathcal{L},\mathcal{R})\Big)^r = \Big(\frac{M_t(\mathcal{L},\mathcal{R}) + m_t(\mathcal{L},\mathcal{R})}{2} - m_t(\mathcal{L},\mathcal{R})\Big)^r \\
&= \frac{\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^r}{2^r}.
\end{aligned}$$

Note that

$$[f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t) = \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^r.$$

It follows with some calculations that

$$\begin{aligned}
&[\omega\circ A_s](\mathcal{L}(t),\mathcal{R}(t))ds = \omega\big(s\mathcal{L}(t) + (1-s)\mathcal{R}(t)\big) \\
&= \frac{\big(M_t(\mathcal{L},\mathcal{R}) - (s\mathcal{L}(t) + (1-s)\mathcal{R}(t))\big)^{p-1}\big(s\mathcal{L}(t) + (1-s)\mathcal{R}(t) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1}}{\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p}} \\
&= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2}s^{p-1}(1-s)^{p-1},
\end{aligned}$$

and so

$$\begin{aligned}
&\int_0^1 h(s)[\omega\circ A_s](\mathcal{L}(t),\mathcal{R}(t))ds = \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2}\int_0^1 s^{k+p-1}(1-s)^{p-1}ds \\
&= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2}\beta(k+p,p).
\end{aligned}$$

Finally by the use of above results and Theorem 1.2, we obtain that

$$\begin{aligned}
&\frac{1}{2(\frac{1}{2})^k}\cdot\frac{\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^r}{2^r}\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1}\beta(p,p) \\
&\le \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p+r-1}\beta(p,p+r) \\
&\le \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{r+1}\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2}\beta(k+p,p),
\end{aligned}$$

which implies the following inequalities related to beta function:

$$2^{k-r-1}\beta(p,p) \le t\beta(p,p+r) + (1-t)2^{k-r-1}\beta(p,p) \le \beta(p,p+r) \le \beta(k+p,p), \tag{2.4}$$

for $t\in[0,1]\setminus\{\frac{1}{2}\}$, $0\le k\le 1$ and $r\in[1,\infty)$.

Remark 2.1. For the case that $f(x) = \big(M_t(\mathcal{L},\mathcal{R}) - x\big)^r$, with the same argument as above we recapture (2.4) because of the fact $\beta(p,p+r) = \beta(p+r,p)$.

In special case if we set $k = 1$ and $t = 0, 1$, we get

$$\frac{1}{2^r}\beta(p,p) \leq \beta(p+r,p) \leq \beta(1+p,p) = \frac{1}{2}\beta(p,p), \tag{2.5}$$

for $p > 0$ and $r \in [1,\infty)$. From (2.5) and the characterization $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain that

$$\frac{1}{2^r} \leq \frac{\Gamma(2p)\Gamma(p+r)}{\Gamma(p)\Gamma(2p+r)} \leq \frac{1}{2},$$

for $p > 0$ and $r \in [1,\infty)$. In more special case for any $p > 0$, we have the following result:

$$\frac{1}{2}\Gamma(p)\Gamma(2p+1) = \Gamma(2p)\Gamma(p+1).$$

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