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NON-LINEAR MAPS PRESERVING THE PSEUDO SPECTRUM OF OPERATOR PRODUCTS

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space H. Fix $\epsilon > 0$ and $T \in B(H)$, let $\sigma_\epsilon(T)$ denote the ϵ-pseudo spectrum of T. In this note, we show if the surjective map φ on $B(H)$ satisfies

$$\sigma_\epsilon(TS - ST^*) = \sigma_\epsilon(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*), \ (T, S \in B(H)),$$

then there exists a unitary operator $U \in B(H)$ such that $\varphi(T) = \mu UTU^*$ for every $T \in B(H)$, where $\mu \in \{-1, 1\}$.

1. Introduction

Throughout this paper, $B(H)$ stands for the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $(H, \langle , \rangle)$ and its unit will be denoted by I. Let $B_s(H)$ (resp. $B_a(H)$) be the real linear space of all self-adjoint (resp. anti-self-adjoint) operators in $B(H)$. For an operator $T \in B(H)$, the adjoint and the spectrum of T are denoted by T^* and $\sigma(T)$, respectively. For $\epsilon > 0$, the ϵ-pseudo spectrum of T, $\sigma_\epsilon(T)$, is defined by $\sigma_\epsilon(T) = \cup\{\sigma(T + A) : A \in B(H), \|A\| \leq \epsilon\}$ and coincides with the set

$$\{\lambda \in \mathbb{C} : \|(\lambda I - T)^{-1}\| \geq \epsilon^{-1}$$

with the convention that $\|(\lambda I - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$. It is a compact subset of $\mathbb{C}$ and contains $\sigma(T)$, the spectrum of T. Unlike the spectrum,

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which is a purely algebraic concept, the ϵ-pseudo spectrum depends on the norm. The ϵ-pseudo spectral radius of T, $r_\epsilon(T)$, is given by

$$r_\epsilon(T) = sup\{|\lambda| : \lambda \in \sigma_\epsilon(T)\}.$$

Pseudo spectra are a useful tool for analyzing operators, furnishing a lot of information about the algebraic and geometric properties of operators and matrices. They play a very natural role in numerical computations, especially in those involving spectral perturbations. The book [5] gives an extensive account of the pseudo spectra, as well as investigations and applications in numerous fields.

Linear preserver problems, in the most general setting, demands the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant. In all cases that have been studied by now, the maps are either supposed to be linear, or proved to be so. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [4], who characterized linear maps that preserve the determinant of matrices. The study of nonlinear pseudo spectrum preserver problems attracted the attention of a number of authors. Cui et al. [2, Theorem 3.3] characterized maps on $M_n(\mathbb{C})$ that preserve the ϵ-pseudo spectrum of the usual product of matrices. They proved that a map φ on $M_n(\mathbb{C})$ satisfies

$$\sigma_\epsilon(\varphi(T)\varphi(S)) = \sigma_\epsilon(TS)\ (T, S \in M_n(\mathbb{C}))$$

if and only if there exist a scalar $c = \pm 1$ and a unitary matrix $U \in M_n(\mathbb{C})$ such that $\varphi(T) = cUTU^*$ for all $T \in M_n(\mathbb{C})$. This result was extended to the infinite dimensional case by Cui et al. [3, Theorem 4.1], where the authors showed that a surjective map φ on $B(H)$ preserves the ϵ-pseudo spectrum of the product of operators if and only if it is a unitary similarity transform up to a scalar $c = \pm 1$. The aim of this note is to characterize mappings on $B(H)$ that preserve the ϵ-pseudo spectral of the skew Lie product "$[T, S]_* = TS - ST^*$" of operators. For two nonzero vectors x and y in H, let $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = \langle z, y\rangle\, x, \qquad \forall\ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i, y_i \in H$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.

In the following proposition, we collects some known properties of the ϵ-pseudo spectrum which are needed in the proof of the main result.

Let $\epsilon > 0$ be arbitrary and $D(0, \epsilon) = \{\mu \in C : |\mu - a| < \epsilon\}$, where $a \in C$.

Proposition 1.1. *(See [3, 5].)*
Let $\alpha > 0$ and let $T \in B(H)$.

(1) $\sigma(T) + D(0, \epsilon) \subseteq \sigma_\epsilon(T)$.
(2) If T *is normal, then* $\sigma_\epsilon(T) = \sigma(T) + D(0, \epsilon)$.
(3) For any $\alpha \in \mathbb{C}, \sigma_\epsilon(T + \alpha I) = \alpha + \sigma_\epsilon(T)$.
(4) For any nonzero $\alpha \in \mathbb{C}, \sigma_\epsilon(\alpha T) = \alpha\sigma_{\frac{\epsilon}{|\alpha|}}(T)$.
(5) For any $\alpha \in \mathbb{C}$, *we have* $\sigma_\epsilon(T) = D(\alpha, \epsilon)$ *if and only if* $T = \alpha I$.
(6) If $\alpha \in \mathbb{C}$ *is a nonzero scalar, then* $\sigma_\epsilon(T) = D(0, \epsilon) \cup D(\alpha, \epsilon)$ *if and only if there exists a nontrivial orthogonal projection* $P \in P(H)$ *such that* $T = \alpha P$.

2. Main Results

The following lemma is a key tool for the proof of main result and describes the spectrum of the skew Lie product $[x \otimes y, T]_*$ for any nonzero vectors $x, y \in H$ and operator $T \in B(H)$.

Lemma 2.1. *(See* [1, Lemma 2.1]*.) For any nonzero vectors* $x, y \in H$ *and* $T \in B(H)$, *set*

$$\Delta_T(x, y) = (\langle Tx, y\rangle + \langle Ty, x\rangle)^2 - 4\|x\|^2 \left\langle T^2y, y\right\rangle$$

and

$$\Lambda_T(x, y) = (\langle x, Ty\rangle + \langle Tx, y\rangle)^2 - 4\|x\|^2 \langle Tx, Ty\rangle$$

Then
(1) $\sigma([x \otimes y, T]_*) = \frac{1}{2}\{0, \langle Tx, y\rangle - \langle Ty, x\rangle \pm \sqrt{\Delta_T(x, y)}\ \}$,
(2) $\sigma([T, x \otimes y]_*) = \frac{1}{2}\{0, \langle Tx, y\rangle - \langle x, Ty\rangle \pm \sqrt{\Lambda_T(x, y)}\ \}$.

Corollary 2.2. *(See* [1, Lemma 2.1]*.) For any* $x \in H$ *and* $T \in B(H)$, *we have*

$$\sigma(T(x \otimes x) + (x \otimes x)T) = \{0, \langle Tx, x\rangle \pm \sqrt{\langle T^2x, x\rangle}\ \}.$$

The followig theorem is the main result of this paper.

Theorem 2.3. *Suppose that a surjective map* $\varphi : B(H) \to B(H)$ *satisfies*

$$\sigma_\epsilon(TS - ST^*) = \sigma_\epsilon(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*),\ (T, S \in B(H)).$$

Then there exists a unitary operator $U \in B(H)$ *such that* $\varphi(T) = \mu UTU^*$ *for every* $T \in B(H)$, *where* $\mu \in \{-1, 1\}$.

Proof. The proof of it will be completed after checking several claims.

Claim 1. φ is injective and $\varphi(0) = 0$.

Claim 2. φ preserves self-adjoint and anti-self adjoint operators in both directions.

Claim 3. $\varphi(iI) = iI$.

Claim 4. The result in the theorem holds.

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